CROSSED PRODUCT C*-ALGEBRAS BY FINITE GROUP ACTIONS WITH THE PROJECTION FREE TRACIAL ROKHLIN PROPERTY

DAWN ARCHEY

ABSTRACT. In this paper we introduce an analog of the tracial Rokhlin property, called the *projection free tracial Rokhlin property*, for C^*-algebras which may not have any nontrivial projections. Using this we show that if A is an infinite dimensional stably finite simple unital C^*-algebra with stable rank one, with strict comparison of positive elements, with only finitely many extreme tracial states, and with the property that every 2-quasi-trace is a trace, and if α is an action of a finite group G with the projection free tracial Rokhlin property, then the crossed product $C^*(G, A, \alpha)$ also has stable rank one.

1. INTRODUCTION

The tracial Rokhlin property for finite group actions was introduced in [17] where it was shown to be a sufficient condition on the action for proving that tracial rank zero is preserved when taking crossed products of simple separable unital C^*-algebras. Similarly, the tracial Rokhlin property is a sufficient condition on the action on a large class of C^*-algebras to show that order on projections determined by traces, real rank zero, and stable rank one (when all are present) are preserved by taking crossed products [1]. The examples in [16] provide counter examples to various strengthenings of results in [17]. In [5], open questions about the standard actions of finite subgroups of $SL_2(\mathbb{Z})$ on irrational rotation algebras were successfully addressed using the tracial Rokhlin property. The paper [15] answered questions about the structure of simple higher dimensional noncommutative toruses and the structure of their crossed products by the flip.

Although the tracial Rokhlin property has had many successes, it also has inherent limitations. It is clear from the definition of the tracial Rokhlin property that it guarantees the existence of at least n projections, where n is the order of the group. In fact, the tracial Rokhlin property implies either that the action has the Rokhlin property or that the algebra has property (SP); see Lemma 1.13 in [17]. Therefore, a C^*-algebra with few projections cannot have any action with the tracial Rokhlin property.

Thus in Section 2 we have formulated a projection free generalization of the tracial Rokhlin property called the *projection free tracial Rokhlin property*.

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This generalization replaces the projections with positive elements and Murray-von Neumann equivalence with Cuntz equivalence of positive elements. The increased flexibility of this definition comes from the fact that some C^*-algebras have no non-trivial projections, but every C^*-algebra is generated by its positive elements.

The main result of the paper is Theorem 3.17, namely:

Theorem. *Let A be an infinite dimensional stably finite simple unital C^*-algebra with stable rank one and strict comparison of positive elements. Assume also that every 2-quasi-trace on A is a trace and that A has only finitely many extreme tracial states. Let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Then $C^*(G, A, \alpha)$ has stable rank one.*

In Section 4, we briefly discuss the necessity of some of the conditions in the definition of the projection free tracial Rokhlin property.

In the final section of this paper, we demonstrate that if Z is the Jiang-Su algebra as defined in [7], then the action which interchanges the two copies of Z in $Z \otimes Z$ provides an example of an action with the projection free tracial Rokhlin property. We also note that Z satisfies the other hypotheses of Theorem 3.17. As a corollary, we get that the crossed product by this action has stable rank one. This was previously shown in a completely different way in Theorem 1.1 of [6].

2. THE PROJECTION FREE TRACIAL ROKHLIN PROPERTY

Definition 2.1. *Let x and y be positive elements of a C^*-algebra A. We write $x \precsim y$ if there exist elements r_j in A such that $r_j y r_j^* \to x$ with convergence in norm. In this case we say x is* **(Cuntz) subequivalent** *to y. If $x \precsim y$ and $y \precsim x$, we write $x \sim y$ and say x is* **(Cuntz) equivalent** *to y.*

Definition 2.2. *For $\varepsilon > 0$, let f_ε be given by $f_\varepsilon(t) = 0$ for $0 \le t \le \varepsilon$, by $f_\varepsilon(t) = \varepsilon^{-1}(t - \varepsilon)$ for $\varepsilon \le t \le 2\varepsilon$ and $f_\varepsilon(t) = 1$ for $t \ge 2\varepsilon$.*

It is useful to have alternate formulations of the concept of Cuntz subequivalence. The following proposition is Proposition 2.4 in [20].

Proposition 2.3. *Let f_ε be as in Definition 2.2. Let x, y be positive elements of the unital C^*-algebra A. The following are equivalent:*

(1) $x \precsim y$.
(2) For all $\varepsilon > 0$, there exists $r \in A$ with $f_\varepsilon(x) \le r y r^*$.
(3) There exist elements r_j and s_j of A with $r_j y s_j \to x$.
(4) For all $\varepsilon > 0$, there exists $\delta > 0$ and $r \in A$ such that $f_\varepsilon(x) = r f_\delta(y) r^*$.

Additionally, if A has stable rank 1, then (1)-(4) above are equivalent to:

5. For all $\varepsilon > 0$ there exists a unitary $u \in A$ such that $u f_\varepsilon(x) u^* \in \overline{yAy}$.

The following proposition is useful for determining subequivalence of elements constructed using functional calculus.

Proposition 2.4. *Let f and g be positive functions in $C(X)$ or $C_0(X)$ for some space X.*

(1) If $\{x \in X : f(x) \neq 0\} \subset \{x \in X : g(x) \neq 0\}$, then $f \precsim g$.
(2) Suppose that $f \precsim g$, that $X \subset [0, \infty)$, and that $a \in A$ is a positive self adjoint element of a C^*-algebra A with $\mathrm{sp}(a) \subset X$. Then $f(a) \precsim g(a)$.

Proof. The first part is a comment just before Proposition 2.1 of [20].

For the second part, let $h_j \in C(X)$ be functions such that $h_j g h_j^* \to f$. Then, since functional calculus is a continuous homomorphism, $(h_j g h_j^*)(a) = h_j(a)g(a)h_j^*(a) \to f(a)$. Therefore, $f(a) \precsim g(a)$ by definition. ∎

The following definition appears near the end of section 2 of [2].

Definition 2.5. *Given a normalized 2-quasi-trace τ on A, one may define a map*

$$d_\tau : M_\infty(A)_+ = (\cup_{n=1}^\infty M_n(A))_+ \to \mathbb{R}_+$$

by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}).$$

We say that A has strict comparison (of positive elements) *if $\lim_{n\to\infty} \tau(a^{1/n}) < \lim_{n\to\infty} \tau(b^{1/n})$ for every normalized 2-quasi-trace τ on A, implies $a \precsim b$ for all elements $a, b \in A_+ \setminus \{0\}$.*

Notice that since the definition is already treating $M_\infty(A)$, if A has strict comparison, so does $M_n(A)$ for any positive integer n.

Lemma 2.6. *If A has strict comparison and $c \in A_+$, then $\overline{cAc}$ has strict comparison.*

Proof. Without loss of generality, $\|c\| = 1$. Suppose $a, b \in A_+$ and $\lim_{n\to\infty} \tau(a^{1/n}) < \lim_{n\to\infty} \tau(b^{1/n})$ for every normalized 2-quasi-trace on $\overline{cAc}$. Note that any 2-quasi-trace σ on A restricts to a 2-quasi-trace on $\overline{cAc}$, so for such σ, we have $\lim_{n\to\infty} \sigma(a^{1/n}) < \lim_{n\to\infty} \sigma(b^{1/n})$. Therefore, $a \precsim b$ in A by the strict comparison on A. That means there exist elements $r_j \in A$ such that $r_j b r_j^* \to a$ in norm.

We now must show that for any $\varepsilon > 0$ there exists $s \in \overline{cAc}$ such that $\|sbs^* - a\| < \varepsilon$. Choose J such that if $j \geq J$, then $\|r_J b r_J^* - a\| < \varepsilon/3$. We may assume that $r_J \neq 0$, because $0 \in \overline{cAc}$. Next choose K sufficiently large that $\|c^{1/K} a c^{1/K} - a\| < \varepsilon/3$ and such that $\|c^{1/K} b c^{1/K} - b\| < \frac{\varepsilon}{3\|r_J\|^2}$.

Take $s = c^{1/K} r_J c^{1/K}$. Then compute

$$\begin{aligned}
\|sbs^* - a\| &\leq \|c^{1/K} r_J c^{1/K} b c^{1/K} r_J^* c^{1/K} - c^{1/K} r_J b r_J^* c^{1/K}\| \\
&\quad + \|c^{1/K} r_J b r_J^* c^{1/K} - c^{1/K} a c^{1/K}\| + \|c^{1/K} a c^{1/K} - a\| \\
&\leq \|c^{1/K}\|^2 \|r_J\|^2 \|c^{1/K} b c^{1/K} - b\| \\
&\quad + \|c^{1/K}\|^2 \|r_J b r_J^* - a\| + \|c^{1/K} a c^{1/k} - a\| \\
&< \varepsilon.
\end{aligned}$$

Therefore $a \precsim b$ in $\overline{cAc}$ as well. ∎

The following definition is a projection free analog of Definition 1.2 of [17], it is the promised generalization of the tracial Rokhlin property.

Definition 2.7. *Let A be an infinite dimensional stably finite unital simple C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say α has the* projection free tracial Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ of norm 1, there exist elements $a_g \in A$ of norm one for each $g \in G$ which are mutually orthogonal, satisfy $0 \leq a_g \leq 1$, and such that:*

(1) $\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ *for all* $g, h \in G$.
(2) $\|a_g b - b a_g\| < \varepsilon$ *for all* $g \in G$ *and* $b \in F$.
(3) *With* $a = \sum_{g\in G} a_g$ *we have* $\tau(1-a) < \varepsilon$ *for all* $\tau \in T(A)$.
(4) *With* $a = \sum_{g\in G} a_g$, *the element* $1-a$ *is Cuntz subequivalent to an element of the hereditary subalgebra generated by* x.

The last condition is not needed if A has strict comparison and every 2-quasi-trace is a trace (See section 4).

Lemma 2.8. *If* a_j *are mutually orthogonal elements for* $j = 1, \ldots, n$, *then* $\|\sum_{j=1}^n a_j\| = \max \|a_j\|$. *In particular, if* a *and* a_g *are as in Definition 2.7, then* $\|a\| = \max_{g\in G} \|a_g\| = 1$. *Furthermore, since* a *is positive* $\|a^2\| = \|a\|^2 = 1$.

Proof. Since positive mutually orthogonal elements commute, it is sufficient to prove that for any $n \in \mathbb{N}$, if $f_1, \ldots f_n$ are positive mutually orthogonal continuous functions on X, then $\|\sum_{i=1}^n f_i\| = \max\{\|f_1\|, \ldots, \|f_n\|\}$. However, this statement is obvious since for each $x \in X$ we can have $f_i(x) \neq 0$ for at most one index i. ∎

The following lemma and its corollary are analogs of Lemma 1.5 and Corollary 1.6 of [17].

Lemma 2.9. *Let* A *be a simple infinite dimensional stably finite unital* C^**-algebra. Let* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group with the projection free tracial Rokhlin property. Then* α_g *is outer for every* $g \in G \setminus \{1\}$.

Proof. Suppose u is a unitary and $g \neq 1$. We will show that there is some b such that $\|u^* b u - \alpha_g(b)\| > 1/3$. Apply the projection free tracial Rokhlin property with $\varepsilon = 1/3$, and with $F = \{u\}$ to get mutually orthogonal $a_g \in A$ for each $g \in G$ with $0 \leq a_g \leq 1$ satisfying the properties there. By Lemma 2.8 we have

$$\begin{aligned}
\|\alpha_g(a_h) - u a_h u^*\| &> \|a_{gh} - a_h\| - \|u a_h u^* - a_h\| - \|\alpha_g(a_h) - a_{gh}\| \\
&\geq \max_{k\in G}\{\|a_k\|\} - \varepsilon - \varepsilon \\
&= 1 - 2\varepsilon \\
&= 1/3.
\end{aligned}$$

This completes the proof. ∎

Corollary 2.10. *Let* A *be an infinite dimensional stably finite simple unital* C^**-algebra and let* $\alpha : G \to \mathrm{Aut}(A)$ *be an action of a finite group with the projection free tracial Rokhlin property. Then* $C^*(G, A, \alpha)$ *is simple.*

Proof. In view of Lemma 2.9, this follows from 3.1 of [8]. ∎

Lemma 2.11. *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous with* $f(0) = 0$ *and* $a_1, \ldots, a_n \in A_+$ *are mutually orthogonal, then* $f(\sum_{i=1}^n a_i) = \sum_{i=1}^n f(a_i)$.

Proof. After showing the lemma holds for $f(x) = x^n$, one shows the lemma holds for any f with standard functional calculus techniques. ∎

Lemma 2.12. *Suppose* $f : [0,1] \to \mathbb{R}$ *is continuous. Then for all* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that for any* C^**-algebra* A *and any self-adjoint elements* x *and* y *of* A *with* $\mathrm{sp}(x), \mathrm{sp}(y) \subset [0,1]$ *and* $\|x - y\| < \delta$, *then* $\|f(x) - f(y)\| < \varepsilon$.

Proof. This is another standard functional calculus argument. ∎

Lemma 2.13. *Suppose $f : [0,1] \to \mathbb{R}$ is continuous. Then for all $\varepsilon > 0$, there exists a $\delta > 0$ such that if x is self adjoint in some C^*-algebra D with $\mathrm{sp}(x) \subset [0,1]$ and if $z \in D$ with $\|z\| \leq 1$ and $\|[x,z]\| < \delta$, then $\|[f(x),z]\| < \varepsilon$.*

Proof. We first show that the lemma is true for any monomial $f(t) = t^k$ using induction. Then we complete the proof using a standard functional calculus argument. ∎

Lemma 2.14. *Let τ be a tracial state on A. For all $\varepsilon > 0$, there is a $\delta > 0$ such that if $g : [0,1] \to [0,1]$ is a continuous function satisfying $g(0) = 0$ and $g(t) = 1$ for $t \in [1-\varepsilon, 1]$, and if $a \in A$ with $0 \leq a \leq 1$ and with $\tau(a) > 1-\delta$, then $\tau(1-g(a)) < \varepsilon$. Moreover, we may choose $\delta = \varepsilon^2$.*

Proof. Let μ be the measure on $\mathrm{sp}(a) \subset [0,1]$ obtained from τ. If $\tau(a) > 1-\delta$, then

$$\begin{aligned}
1-\delta &< \tau(a) \\
&\leq (1-\varepsilon)\mu([0,1-\varepsilon]) + 1 \cdot \mu((1-\varepsilon,1]) \\
&= (1-\varepsilon)\mu([0,1-\varepsilon]) + 1 - \mu([0,1-\varepsilon]) \\
&= \mu([0,1-\varepsilon]) - \varepsilon\mu([0,1-\varepsilon]) + 1 - \mu([0,1-\varepsilon]) \\
&= 1 - \varepsilon\mu([0,1-\varepsilon]),
\end{aligned}$$

which implies that

$$\frac{\delta}{\varepsilon} > \mu([0,1-\varepsilon]).$$

Now we compute

$$\tau(1-g(a)) = \int_{[0,1]} (1-g(t))\, d\mu \leq 1 \cdot \mu([0,1-\varepsilon]) < \frac{\delta}{\varepsilon}.$$

So if $\delta \leq \varepsilon^2$ then

$$\tau(1-g(a)) < \frac{\varepsilon^2}{\varepsilon} = \varepsilon.$$

∎

Lemma 2.15. *Suppose A is an infinite dimensional stably finite simple unital C^*-algebra. Suppose G is a finite group. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G with the projection free tracial Rokhlin property. Let $E : C^*(G,A,\alpha) \to A$ be the conditional expectation. Suppose $\tau \in T(C^*(G,A,\alpha))$, then there exists $\sigma \in T(A)$ such that $\tau = \sigma \circ E$.*

Proof. It suffices to show that if $x \in A$ and $g \in G \setminus \{0\}$, then $|\tau(xu_g)| < \varepsilon$ for any $\varepsilon > 0$. Let $\varepsilon > 0$ be given. Let $n = \mathrm{card}(G)$. Without loss of generality, $\|x\| \leq 1$.

Choose δ_1 using Lemma 2.13 with $\frac{\varepsilon}{3n}$ in place of ε and with $t^{1/2}$ in place of f. Choose δ_2 using Lemma 2.12 with $\frac{\varepsilon}{3n}$ in place of ε and with $t^{1/2}$ in place of f. Choose $\delta_3 < \min\{\frac{\varepsilon^2}{18}, \frac{1}{2}\}$. Choose δ_4 using Lemma 2.14 with δ_3 in place of ε. Choose a continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0$ and $g(t) = 1$ for $t \in [1-\delta_3, 1]$. We also require that $\|g - (2t - t^2)\| < \delta_3$. This is possible since

$$\sup\left\{\left\|1 - \left(2t - t^2\right)\right\| : t \in [1-\delta_3, 1]\right\} = \delta_3^2.$$

Apply the projection free tracial Rokhlin property with δ_4 in place of ε and with $F = \{x\}$ to get mutually orthogonal positive elements a_h for each $h \in G$. Set $a = \sum_{h\in G} a_h$. One of the properties satisfied by a is that $\tau(a) > 1 - \delta_4$. By the choice of g and δ_4, this implies $\tau(1 - g(a)) < \delta_3$. By the second requirement on g we now have $\tau((1 - a)^2) = \tau(1 - (2a - a^2)) < \tau(1 - g(a)) + \delta_3 < 2\delta_3$.

By the Cauchy-Schwartz inequality, we have

$$|\tau(xu_g(1 - a))|^2 \leq \tau(u_g xx^* u_g^*)\tau((1 - a)^2) \leq \|x\|^2\tau((1 - a)^2) < 2\delta_3 < \frac{\varepsilon^2}{9}.$$

Therefore, we can conclude

$$|\tau(xu_g(1 - a))| < \varepsilon/3. \tag{2.1}$$

We are now in a position to compute $|\tau(xu_g)|$. We have

$$\begin{aligned}
|\tau(xu_g)| &= \left|\tau(xu_g) - \sum_{h\in G}\tau(xa_h^{1/2}a_{gh}^{1/2}u_g)\right| \\
&\leq |\tau(xu_g) - \tau(xu_g a)| + \left|\sum_{h\in G}\tau(xu_g a_h) - \sum_{h\in G}\tau\left(a_h^{1/2}xu_g a_h^{1/2}\right)\right| \\
&\quad + \left|\sum_{h\in G}\tau\left(a_h^{1/2}xu_g a_h^{1/2}u_g^* u_g\right) - \sum_{h\in G}\tau\left(a_h^{1/2}xa_{gh}^{1/2}u_g\right)\right| \\
&\quad + \left|\sum_{h\in G}\tau\left(a_h^{1/2}xa_{gh}^{1/2}u_g\right) - \sum_{h\in G}\tau\left(xa_h^{1/2}a_{gh}^{1/2}u_g\right)\right| \\
&\leq |\tau\left(xu_g(1 - a)\right)| + 0 + \sum_{h\in G}\left|\tau\left(a_h^{1/2}xu_g a_h^{1/2}u_g^* u_g - a_h^{1/2}xa_{gh}^{1/2}u_g\right)\right| \\
&\quad + \sum_{h\in G}|\tau(a_h^{1/2}xa_{gh}^{1/2}u_g - xa_h^{1/2}a_{gh}^{1/2}u_g)| \\
&< \varepsilon/3 + \sum_{h\in G}\left\|u_g a_h u_g^* - a_{gh}^{1/2}\right\| + \sum_{h\in G}\left\|a_h^{1/2}x - xa_h^{1/2}\right\| \quad \text{by Equation 2.1} \\
&< \varepsilon \quad \text{by the choice of } \delta_1 \text{ and } \delta_2 .
\end{aligned}$$

This completes the proof. ∎

Remark 2.16. *The hypothesis that all 2-quasi-traces are traces appears frequently in what follows. Thus it is worth noting as is done near the end of Section 2 of* [2] *that every exact C^*-algebra satisfies this hypothesis. Whether this is true in general is an open question.*

Lemma 2.17. *Let A be a C^*-algebra with strict comparison. Fix $z \in A$ with $0 \leq z \leq 1$ and $z \neq 0$. If $0 < \varepsilon < \tau(z)$ for every 2-quasi-trace τ, if $g : [0, 1] \to [0, 1]$ is continuous and satisfies $g(0) = 0$ and $g(t) = 1$ for $t \in [1 - \varepsilon, 1]$, and if $a \in A$ with $0 \leq a \leq 1$ and $\tau(a) > 1 - \varepsilon^2$ for every 2-quasi-trace τ, then $1 - g(a) \preccurlyeq z$.*

Proof. We first claim that $(1-g(t))^{1/n} = 1-g_n(t)$ for some continuous g_n satisfying $g_n(0) = 0$ and $g_n(t) = 1$ for $t \in [1-\varepsilon, 1]$. To see this, observe that this is equivalent to saying $(1-g(t))^{1/n} = f_n(t)$ for some continuous function f_n satisfying $f_n(0) = 1$ and $f_n(t) = 0$ for $t \in [1 - \varepsilon, 1]$. But the left-hand side is the composition of

continuous functions, hence continuous, and the left-hand side maps 0 to 1 and $[1-\varepsilon, 1]$ to 0, so the equivalent statement is clear.

By the claim and Lemma 2.14, since each g_n is a function of the same type as g, we have $\tau((1-g(a))^{1/n}) = \tau(1-g_n(t)) < \varepsilon$. So now

$$\tau((1-g(a))^{1/n}) < \varepsilon < \tau(z) < \tau(z^{1/2}) < \tau(z^{1/3}) < \cdots,$$

which implies that

$$\lim_{n\to\infty} \tau((1-g(a))^{1/n}) \le \varepsilon < \tau(z) \le \lim_{n\to\infty} \tau(z^{1/n}),$$

which gives

$$\lim_{n\to\infty} \tau((1-g(a))^{1/n}) < \lim_{n\to\infty} \tau(z^{1/n}).$$

Since A has strict comparison, it follows that $1-g(a) \precsim z$. ∎

3. Stable rank and the projection free tracial Rokhlin property

Lemma 3.1. *Let f be a continuous function on $[0,1]$ with $f(0)=0$. Let $\{e_{g,h}\}$ be a set of matrix units for M_n. Then in $C([0,1])\otimes M_n$, we have $f(t\otimes e_{g,g}) = f(t)\otimes e_{g,g}$.*

Proof. This is proved using standard functional calculus arguments. ∎

The following proposition and proof are very similar to Proposition 3.3.1 of [9].

Proposition 3.2. *The universal C^*-algebra A generated by $\{y_{j,k} : 1 \le j,k \le n\}$ subject to the relations*

(1) $y_{j_1,k_1} y_{j_2,k_2} = \delta_{k_1,j_2} y_{j_1,j_1} y_{j_1,k_2}$,
(2) $y_{j,k}^* = y_{k,j}$,
(3) $y_{1,1} \neq 0$, *and*
(4) $0 \le y_{j,j} \le 1$

is isomorphic to CM_n.

Proof. We identify CM_n as $C_0((0,1])\otimes M_n$. Let $\{e_{j,k}\}$ be an n by n set of matrix units for M_n. Define the map $\phi : A \to CM_n$ by $y_{j,k} \mapsto t\otimes e_{j,k}$. Since the elements $\{t\otimes e_{j,k}\}$ satisfy the relations which $\{y_{j,k}\}$ satisfy, this a well defined homomorphism.

By the Stone-Weierstrass Theorem the elements $\{t\otimes e_{j,k}\}$ generate CM_n.

Consider an irreducible representation $\pi : A \to H$ of these relations. Let $z_{j,k} = \pi(y_{j,k})$. Consider the element $c = z_{1,1}^2 + \cdots + z_{n,n}^2$. For any j and k between 1 and n,

$$z_{j,k}c = z_{j,k}z_{k,k}^2 = z_{j,j}^2 z_{j,k} = c z_{j,k}.$$

Thus c is central in $C^*(\{z_{j,k}\}_{1\le j,k\le n})$. Because π is irreducible, this implies that c is a scalar multiple of the identity. That is, for some $\gamma\in[0,1]$, we have $c=\gamma I$.

If $\gamma = 0$, then $c=0$. In this case, given l and k with $1\le l,k\le n$, we have

$$0 = c = \sum_{j=1}^{l-1} z_{j,j}^2 + \sum_{j=l+1}^{n} z_{j,j}^2 + z_{l,k}z_{l,k}^*.$$

Note that this sum consists entirely of positive elements and yet adds to zero, therefore each item in the sum is zero. In particular $z_{l,k}z_{l,k}^* = 0$ which implies $z_{l,k}=0$. Therefore, if $\gamma=0$, then $z_{l,k}$ is the image of $t\otimes e_{l,k}$ under the zero representation of CM_n.

If $\gamma > 0$, then γ^{-1} is defined. Note that $\gamma z_{j,j}^2 = c z_{j,j}^2 = z_{j,j}^4$. This implies that $\gamma^{-1} z_{j,j}^2$ is a projection for every j. From this we can also conclude that $\gamma^{-1/2} z_{j,j}$ is a projection. Next we observe that $\gamma^{-1/2} z_{j,k}$ is a partial isometry between $\gamma^{-1/2} z_{j,j}$ and $\gamma^{-1/2} z_{k,k}$. Therefore, the elements $\gamma^{-1/2} z_{j,k}$ satisfy the relations for a set of matrix units for M_n.

Up to unitary equivalence, $H = \mathbb{C}^n$ and $z_{j,k} = \gamma^{1/2} e_{j,k}$. These are the images of $\{t \otimes e_{j,k}\}$ under evaluation at $\gamma^{1/2}$. Thus by Lemma 3.2.2 of [9] we are done. ∎

The following lemma guarantees the existence of elements of $C^*(G, A, \alpha)$ which satisfy the cone relations above (conditions 1–4), approximately respect the action of G (conditions 5 and 6), and are near elements produced using the projection free tracial Rokhlin property (conditions 7–12).

Lemma 3.3. *Suppose A is an infinite dimensional stably finite unital simple C^*-algebra. Let $\varepsilon > 0$ and let $F \subset A$ be a finite set. Suppose G is a finite group and $\alpha : G \to \mathrm{Aut}(A)$ is an action of G on A with the projection free tracial Rokhlin property. Then there exist δ, with $\delta < \varepsilon$, positive elements $a_g \in A$ for each $g \in G$, and elements $Y_{g,h} \in C^*(G, A, \alpha)$ for each $g, h \in G$ such that for $g, h, j, k \in G$ we have*

(1) $Y_{j,k} Y_{g,h} = \delta_{k,g} Y_{j,j} Y_{j,h}$.
(2) $Y_{j,k}^* = Y_{k,j}$.
(3) $Y_{1,1} \neq 0$, *where 1 is the identity of G.*
(4) $0 \leq Y_{1,1} \leq 1$.
(5) $\|u_k Y_{j,l} - Y_{kj,l}\| < \varepsilon$.
(6) $\|Y_{j,g} u_k^* - Y_{j,kg}\| < \varepsilon$.
(7) $\|Y_{j,j} - a_j\| < \varepsilon$.
(8) $Y_{1,1} \in A$.
(9) $\|Y_{j,j} b - b Y_{j,j}\| < 2\varepsilon \|b\| + \varepsilon$ *for any $b \in F$.*
(10) $\|\alpha_j(a_k) - a_{jk}\| < \delta$.
(11) $\|a_j b - b a_j\| < \delta$ *for all $b \in F$.*
(12) *With $\sum_{g \in G} a_g$ we have $\tau(1 - a) < \delta$ for all $\tau \in T(A)$.*

Proof. First observe that (5) and (6) are equivalent by taking adjoints, so we will only prove (5). In order to show (5), it suffices to show $\|u_j Y_{1,k} - Y_{j,k}\| < \varepsilon/2$ and $\|u_{j^{-1}} Y_{j,k} - Y_{1,k}\| < \varepsilon/2$.

We will proceed by induction on the matrix size of the cone, showing at each stage that all the relations are satisfied.

First we work on CM_2. Let 1 be the identity of G and let $g \in G$ be a fixed non identity element. Let $\varepsilon > 0$ be given. Choose δ_0 with $0 < \delta_0 < \varepsilon$ such that if x and y are positive elements of norm less than or equal to one in any C^*-algebra, and if $\|x - y\| < \delta_0$, then $\|x^{1/2} - y^{1/2}\| < \varepsilon/4$. Apply the projection free tracial Rokhlin property with δ_0 in place of ε and with F and x as given to get a_j for each group element $j \in G$. Properties (10), (11), and (12) are true by the definition of

the projection free tracial Rokhlin property. Define

$$\begin{aligned} y_{1,1} &= (a_1^{1/2} u_g^* a_g u_g a_1^{1/2})^{1/2} \\ y_{g,g} &= (a_g^{1/2} u_g a_1 u_g^* a_g^{1/2})^{1/2} \\ y_{1,g} &= a_1^{1/2} u_g^* a_g^{1/2} \\ y_{g,1} &= a_g^{1/2} u_g a_1^{1/2}. \end{aligned}$$

Using the fact that a_1 and a_g are mutually orthogonal, it is easy to check that properties (1), (2), and (3) of the statement are satisfied. For (4) we recall from the definition of the projection free tracial Rokhlin property that $0 \leq a_j \leq 1$ for each $j \in G$. This implies
$0 \leq a_1^{1/2} u_g^* a_g u_g a_1^{1/2} \leq a_1 \leq 1$. Therefore, $0 \leq y_{1,1} \leq 1$. Similarly, $0 \leq a_g^{1/2} u_g a_1 u_g^* a_g^{1/2} \leq a_g \leq 1$. Therefore, $0 \leq y_{g,g} \leq 1$.

To show (5), we use

$$\|a_g - u_g a_1 u_g^*\| < \delta_0$$

to compute,

$$\|u_g y_{1,g} - y_{g,g}\| \leq \|u_g a_1^{1/2} u_g^* a_g^{1/2} - a_g\| + \|a_g - (a_g^{1/2} u_g a_1 u_g^* a_g^{1/2})^{1/2}\| < \varepsilon/2.$$

Similarly, $\|u_g y_{1,1} - y_{g,1}\| < \varepsilon/2$. But now

$$\|u_{g^{-1}} y_{g,1} - y_{1,1}\| = \|u_g u_{g^{-1}} y_{g,1} - u_g y_{1,1}\| < \varepsilon/2$$

and

$$\|u_g^{-1} y_{g,g} - y_{1,g}\| = \|u_g u_{g^{-1}} y_{g,g} - u_g y_{1,g}\| < \varepsilon/2.$$

Next we show that (7) holds. By the choice of δ_0, we have

$$\|a_1^{1/2} u_g^* a_g u_g a_1^{1/2} - a_1^2\| \leq \|u_g^* a_g u_g - a_1\| < \delta_0,$$

which implies

$$\|y_{1,1} - a_1\| = \|(a_1^{1/2} u_g^* a_g u_g a_1^{1/2})^{1/2} - a_1\| < \varepsilon/2.$$

Similarly, $\|y_{g,g} - a_g\| < \varepsilon/2$.

For property (8), we note that $u_g^* a_g u_g = \alpha_{g^{-1}}(a_g) \in A$, so $y_{1,1} \in A$.

Next we show (9). For any $b \in F$ and $j = 1$ or $j = g$, we have

$$\|y_{j,j} b - b y_{j,j}\| \leq \|y_{j,j} b - a_j b\| + \|a_j b - b a_j\| + \|b a_j - b y_{j,j}\| < \varepsilon\|b\| + \delta_0 + \varepsilon\|b\| < 2\varepsilon\|b\| + \varepsilon.$$

For the purposes of induction it is helpful to have one more property, namely, that $y_{j,k}$ for $j, k \in \{1, g\}$ are each orthogonal to a_m for all $m \in G \setminus \{1, g\}$. This is clear since $a_j a_m = 0$ if $j \neq m$. This completes the base case.

From now on let the elements of G be called $1, \ldots, j, \ldots, n$ instead of $g_1, \ldots g_n$ to avoid an excess of double subscripts. In order to avoid confusion, 1 will be the identity of G.

Now suppose that for any $\varepsilon_1 > 0$, finite $F \subset A$, and positive x of norm 1, there exists a positive number $\delta = \delta(\varepsilon, m) < \varepsilon$ such that if there exist elements $z_{j,k} \in C^*(G, A, \alpha)$ for $1 \leq j, k \leq m$ and $a_g \in A$ for $g \in G$ such that

(1) $z_{j,k} z_{l,h} = \delta_{k,l} z_{j,j} z_{j,h}$ for $1 \leq j, k, l, h \leq m$,
(2) $z_{j,k}^* = z_{k,j}$ for $1 \leq j, k \leq m$,
(3) $z_{1,1} \neq 0$,
(4) $0 \leq z_{1,1} \leq 1$,
(5) $\|u_k z_{j,l} - z_{kj,l}\| < \varepsilon_1$ if $j, k, l, kj \leq m$,

(6) $\|z_{j,l}u_k^* - z_{j,kl}\| < \varepsilon_1$ if $j, k, l, kl \leq m$,
(7) $\|z_{j,j} - a_j\| < \varepsilon_1$ if $1 \leq j \leq m$,
(8) $z_{1,1} \in A$,
(9) $\|z_{j,j}b - bz_{j,j}\| < 2\varepsilon_1\|b\| + \varepsilon_1$,
(10) $\|\alpha_g(a_h) - a_{gh}\| < \delta$ for all $g, h \in G$,
(11) $\|a_g b - b a_g\| < \delta$ for all $b \in F$ and all $g \in G$,
(12) With $\sum_{g\in G} a_g$ we have $\tau(1-a) < \delta$ for all $\tau \in T(A)$, and
(13) $z_{j,k}a_l = a_l z_{j,k} = 0$ if $1 \leq j, k \leq m$ and $m+1 \leq l \leq n$.

Given any $\varepsilon > 0$ we wish to show we can produce $\delta(\varepsilon, m+1) < \varepsilon$ and elements $y_{j,k}$ and a_g which satisfy the above properties for $1 \leq j, k \leq m+1$, for all $g \in G$ and with ε in place of ε_1 above. Without loss of generality, $\varepsilon < 1$.

Let $0 < \delta_0 < \varepsilon/192$. Choose δ_1 so that if x and y are positive elements with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| < \delta_1$, then $\|x^{1/2} - y^{1/2}\| < \delta_0$. Without loss of generality, $\delta_1 < \delta_0$. Then choose $\delta_2 > 0$ such that if x and y are positive elements with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x - y\| < \delta_2$, then $\|x^{1/2} - y^{1/2}\| < \delta_1/8$. Choose $\delta_3 = \min\{\frac{\varepsilon}{32}, \frac{\delta_1}{4}\}$. Now choose $0 < \delta_4 < \min\{\delta_2, \delta_1/4\}$.

Define a continuous function f to be zero on $[0, \delta_4]$, one at $t = 1$, and linear on $[\delta_4, 1]$. Define a continuous function g to be zero at $t = 0$, one on $[\delta_4, 1]$, and linear on $[0, \delta_4]$. Notice that $\|f(t) - t\| < \delta_4$ and that $fg = f$.

Choose a polynomial p in $C([0,1])$ with $\|p - t^{1/2}\| < \delta_3/3$ and $p(0) = 0$. Write $p(t) = \sum_{m=1}^d b_m t^m$. Let $\lambda_p = \sum_{m=1}^d |b_m|$. Suppose $\psi : CM_n \to B$ is a homomorphism to a C^*-algebra B and $u \in B$ is a unitary satisfying

$$\|u\psi(t \otimes e_{1,k}) - \psi(t \otimes e_{j,k})\| < \frac{\delta_3}{3\lambda_p}.$$

Then

$$\begin{aligned}
&\|u\psi(t^{1/2} \otimes e_{1,k}) - \psi(t^{1/2} \otimes e_{j,k})\| \\
&\leq \|t^{1/2} - p\| + \sum_{m=1}^d \|u\psi(b_m t^m \otimes e_{1,k}) - \psi(b_m t^m \otimes e_{j,k})\| + \|p - t^{1/2}\| \\
&< 2\delta_3/3 + \sum_{m=1}^d |b_m| \|u\psi(t \otimes e_{1,k})\psi(t^{m-1} \otimes e_{k,k}) - \psi(t \otimes e_{j,k})\psi(t^{m-1} \otimes e_{k,k})\| \\
&\leq 2\delta_3/3 + \sum_{m=1}^d |b_m| \frac{\delta_3}{3\sum_{m=1}^d |b_m|} \\
&= \delta_3.
\end{aligned} \tag{3.1}$$

This implies

$$\|u\psi(f^{1/2} \otimes e_{1,k}) - \psi(f^{1/2} \otimes e_{j,k})\| < 2\delta_4 + \delta_3. \tag{3.2}$$

Choose $0 < \delta_5 < \min\{\frac{\delta_1}{2}, \frac{\varepsilon}{32}, \frac{\delta_2}{4}, \frac{\delta_3}{3\lambda_p}\}$. Let $0 < \delta_6 < \min\{\varepsilon/48, \delta_1/2, \delta_5\}$. Apply the induction hypothesis with δ_6 in place of ε_1 to get elements $z_{j,k}$ and a_g which satisfy the fifteen properties above. Thus $\delta(\delta_6, m) < \delta_6$ so we may assume the projection free tracial Rokhlin property was applied with a number smaller than δ_6 in place of ε. Once again, properties (10), (11), and (12) are satisfied by the definition of the projection free tracial Rokhlin property.

These elements $z_{j,k}$ allow us to define a homomorphism $\phi : CM_m \to C^*(G, A, \alpha)$ by $(t \otimes e_{j,k}) \mapsto z_{j,k}$. Let $s_j = \phi(f^{1/2} \otimes e_{j,1})$ for $j = 1, \ldots, m$. For $1 \le j \le m$, set

$$\begin{aligned} y_{m+1,m+1} &= (a_{m+1}^{1/2} u_{m+1} s_1^2 u_{m+1}^* a_{m+1}^{1/2})^{1/2}, \\ y_{j,m+1} &= s_j u_{m+1}^* a_{m+1}^{1/2}, \\ y_{m+1,j} &= a_{m+1}^{1/2} u_{m+1} s_j^*, \\ y_{j,j} &= (s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^{1/2}, \\ y_{j,k} &= y_{j,j} \phi(g \otimes e_{j,k}). \end{aligned}$$

Before we start to prove that these elements satisfy the cone relations, we make some observations. Notice that $s_j \phi(g \otimes e_{k,l}) = \phi(f^{1/2} g \otimes e_{j,1} e_{k,l}) = \phi(f^{1/2} \otimes \delta_{1,k} e_{j,l})$. Also,

$$\phi(g \otimes e_{k,l}) s_j = \phi(f^{1/2} \otimes \delta_{l,j} e_{k,1})$$

which equals s_k if $j = l$.

Notice that

$$(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^d \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k})(s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*)^d$$

for any positive integer d. Therefore, by the continuity of functional calculus, for any continuous function f with $f(0) = 0$ we have

$$f(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*) \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k}) f(s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*).$$

In particular we have

$$(s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^{1/2} \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k})(s_k u_{m+1}^* a_{m+1} u_{m+1} s_k^*)^{1/2}.$$

Therefore,

$$y_{j,j} \phi(g \otimes e_{j,k}) = \phi(g \otimes e_{j,k}) y_{k,k}. \tag{3.3}$$

Similarly, since $s_j \phi(g \otimes e_{j,j}) = \phi(f^{1/2} \otimes e_{1,j}) \phi(g \otimes e_{j,j}) = \phi(f^{1/2} \otimes e_{1,j}) = s_j$, we conclude,

$$y_{j,j} \phi(g \otimes e_{j,j}) = y_{j,j}. \tag{3.4}$$

Now we check property (1). For this portion of the proof assume that $1 \le i, j, k, l, \le m$. It is easy to see that $y_{j,m+1} y_{m+1,j} = y_{j,j}^2$ and that $y_{m+1,j} y_{j,m+1} = y_{m+1,m+1}^2$. Next we see

$$\begin{aligned} y_{j,m+1} y_{m+1,k} &= s_j u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,k}) \\ &= y_{j,j} y_{j,j} \phi(g \otimes e_{j,k}) \\ &= y_{j,j} y_{j,k} \end{aligned}$$

Since $j \le m$, using the fifteenth property of the induction hypothesis at the second step, we have

$$\begin{aligned} y_{j,j} y_{m+1,m+1} &= (s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^*)^{1/2} (a_{m+1}^{1/2} u_{m+1}) \phi(f \otimes e_{1,1} u_{m+1}^* a_{m+1}^{1/2})^{1/2} \\ &= 0. \end{aligned}$$

Now suppose that $j \neq k$. Then

$$\begin{aligned} y_{j,k}y_{l,l} &= y_{j,j}\phi(g \otimes e_{j,k})y_{l,l} \\ &= \delta_{k,l}y_{j,j}\phi(g \otimes e_{j,l})y_{l,l} \\ &= \delta_{k,l}y_{j,j}y_{j,j}\phi(g \otimes e_{j,l}) \qquad \text{by Equation 3.3} \\ &= \delta_{k,l}y_{j,j}y_{j,l}. \end{aligned}$$

If $j \neq k$ we also have $y_{j,j}y_{k,k} = 0$ since $s_j^* s_k = 0$.

Now if $k \neq i$, and $j \neq k$ and $i \neq l$, we have $y_{j,k}y_{i,l} = y_{j,j}\phi(g \otimes e_{j,k})\phi(g \otimes e_{i,l})y_{l,l} = 0$. On the other hand, if $k = i$, but $j \neq k$ and $k \neq l$ we get $y_{j,k}y_{i,l} = y_{j,j}\phi(g^2 \otimes e_{j,l})y_{l,l}$. This shows (1).

For (2), note equation 3.3 implies $y_{j,k}^* = y_{k,j}$ for $1 \leq j,k \leq m$. The rest of the adjoint conditions required for (2) are clear from the definitions of the elements.

Next we show (5) by checking the various cases as we did for (1). However we begin by computing some useful estimates. Using $\|f(t) - t\| < \delta_4 < \delta_2$ and $\|\phi(t \otimes e_{1,1}) - a_1\| < \delta_5 < \delta_2$ for the last step, we compute

$$\begin{aligned} \|y_{1,1}^2 - a_1^2\| &\leq 2\|\phi(f^{1/2} \otimes e_{1,1}) - a_1^{1/2}\| + \|u_{m+1}^* a_{m+1} u_{m+1} - a_1\| \\ &\leq 2\|\phi(f^{1/2} \otimes e_{1,1}) - \phi(t^{1/2} \otimes e_{1,1})\| + 2\|\phi(t^{1/2} \otimes e_{1,1}) - a_1^{1/2}\| + \delta_6 \\ &\leq \delta_1. \end{aligned}$$

By the choice of δ_1, this implies that

$$\|y_{1,1} - a_1\| < \delta_0. \tag{3.5}$$

Using the facts that $\|a_1 - z_{1,1}\| < \delta_5 < \delta_2$ and $\|t - f\| < \delta_4 < \delta_2$ we see that

$$\|a_1^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| \leq \|a_1^{1/2} - z_{1,1}^{1/2}\| + \|z_{1,1}^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| < \delta_1/4. \tag{3.6}$$

Additionally, since $\|u_j a_k u_j^* - a_{jk}\| < \delta_6 < \delta_1$ for $1 \leq j,k \leq m$, we have

$$\|u_j a_k^{1/2} u_j^* - a_{jk}^{1/2}\| < \delta_0. \tag{3.7}$$

Note that

$$\|u_j\phi(t \otimes e_{1,k}) - \phi(t \otimes e_{j,k})\| = \|u_j z_{1,k} - z_{j,l}\| < \delta_5 < \frac{\delta_3}{3\lambda_p}.$$

Thus by Equation 3.1 we have $\|u_j\phi(t^{1/2} \otimes e_{1,k}) - \phi(t^{1/2} \otimes e_{j,k})\| < \delta_3$. Therefore, by Equation 3.2,

$$\|u_j\phi(f^{1/2} \otimes e_{1,k}) - \phi(f^{1/2} \otimes e_{j,k})\| < 2\delta_4 + \delta_3. \tag{3.8}$$

In particular, $\|u_j\phi(f^{1/2} \otimes e_{1,k}) - \phi(f^{1/2} \otimes e_{j,k})\| < \varepsilon/16$.

Additionally,

$$\begin{aligned} \|u_j\phi(g \otimes e_{1,k}) - \phi(g \otimes e_{j,k})\| &= \|u_j\phi(f^{1/2} \otimes e_{1,k})\phi(g \otimes e_{k,k}) - \phi(f^{1/2} \otimes e_{j,k})\phi(g \otimes e_{k,k})\| \\ &< \varepsilon/16. \end{aligned} \tag{3.9}$$

Now

$$\begin{aligned}\left\|\left(u_j y_{1,1} u_j^*\right)^2 - y_{j,j}^2\right\| &= \|u_j y_{1,1}^2 u_j^* - y_{j,j}^2\| \\ &\leq \|u_j \phi(f^{1/2} \otimes e_{1,1}) u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,1}) u_j^* \\ &\qquad - u_j \phi(f^{1/2} \otimes e_{1,1}) u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,j})\| \\ &\quad + \|u_j \phi(f^{1/2} \otimes e_{1,1}) u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,j}) \\ &\qquad - \phi(f^{1/2} \otimes e_{j,1}) u_{m+1}^* a_{m+1} u_{m+1} \phi(f^{1/2} \otimes e_{1,j})\| \\ &\leq 2\|u_j \phi(f^{1/2} \otimes e_{1,1}) - \phi(f^{1/2} \otimes e_{j,1})\| \\ &< 2\delta_4 + \delta_3 \\ &< \delta_1.\end{aligned}$$

Therefore,

$$\|u_j y_{1,1} u_j^* - y_{j,j}\| < \delta_0. \tag{3.10}$$

We now begin to check the cases for (5). If $j = m+1$, but $l \neq m+1$, and $l \neq 1$, then

$$\begin{aligned}\|u_j y_{1,l} - y_{j,l}\| &= \|u_{m+1} y_{1,1} \phi(g \otimes e_{1,l}) - a_{m+1}^{1/2} u_{m+1} s_l^*\| \\ &= \|(s_1 u_{m+1}^* a_{m+1} u_{m+1} s_1)^{1/2} \phi(g \otimes e_{1,l}) - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_l^*\| \\ &\leq \|(s_1 u_{m+1}^* a_{m+1} u_{m+1} s_1)^{1/2} - a_1\| \\ &\quad + \|a_1^{1/2} \phi(g \otimes e_{1,l}) - \phi(f^{1/2} \otimes e_{1,1}) \phi(g \otimes e_{1,l})\| \\ &\quad + \|a_1^{1/2} - u_{m+1}^* a_{m+1}^{1/2} u_{m+1}\| \\ &\leq \delta_0 + \delta_1/4 + \delta_0 \quad \text{by Equations 3.5, 3.6, and 3.7} \\ &< 3\delta_0 \\ &< \varepsilon/2.\end{aligned}$$

Next suppose that $j = m+1$ and $l = 1$. Then, using Equation 3.5 in the third to last step and Equations 3.6 and 3.7 in the second to last step we see that

$$\begin{aligned}\|u_j y_{1,l} - y_{j,l}\| &= \|y_{1,1} - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_1\| \\ &\leq \|y_{1,1} - a_1\| + \|a_1 - a_1^{1/2} s_1\| + \|a_1^{1/2} s_1 - u_{m+1}^* a_{m+1}^{1/2} u_{m+1} s_1\| \\ &< \delta_0 + \|a_1^{1/2} - \phi(f^{1/2} \otimes e_{1,1})\| + \|a_1^{1/2} - u_{m+1}^* a_{m+1}^{1/2} u_{m+1}\| \\ &\leq \delta_0 + 2\delta_0 + \delta_0 \\ &< \varepsilon/2.\end{aligned}$$

Now let $j = l = m+1$. We use $\|u_{m+1} a_1 u_{m+1}^* - a_{m+1}\| < \delta_5 < \delta_1/2$, the estimate $\|\phi(t \otimes e_{1,1}) - a_1\| < \delta_5$, and $\|t - f(t)\| < \delta_4$ for the second to last step to

get

$$
\begin{aligned}
&\|u_j y_{1,l} - y_{j,l}\| \\
&= \left\| u_{m+1}\phi(f^{1/2}\otimes e_{1,1})u_{m+1}^* a_{m+1}^{1/2} - \left(a_{m+1}^{1/2} u_{m+1}\phi(f\otimes e_{1,1})u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} \right\| \\
&\leq \left\| u_{m+1}\phi\left(f^{1/2}\otimes e_{1,1}\right) u_{m+1}^* a_{m+1}^{1/2} - u_{m+1} a_1^{1/2} u_{m+1}^* a_{m+1}^{1/2} \right\| \\
&\quad + \left\| u_{m+1} a_1^{1/2} u_{m+1}^* a_{m+1}^{1/2} - a_{m+1} \right\| \\
&\quad + \left\| a_{m+1} - \left(a_{m+1}^{1/2} u_{m+1} a_1 u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} \right\| \\
&\quad + \left\| \left(a_{m+1}^{1/2} u_{m+1} a_1 u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} - \left(a_{m+1}^{1/2} u_{m+1}\phi\left(f\otimes e_{1,1}\right) u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} \right\| \\
&\leq \left\| \left(a_{m+1}^{1/2} u_{m+1} a_1 u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} - \left(a_{m+1}^{1/2} u_{m+1}\phi\left(t\otimes e_{1,1}\right) u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} \right\| \\
&\quad + \left\| \left(a_{m+1}^{1/2} u_{m+1}\phi\left(t\otimes e_{1,1}\right) u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} - \left(a_{m+1}^{1/2} u_{m+1}\phi\left(f\otimes e_{1,1}\right) u_{m+1}^* a_{m+1}^{1/2}\right)^{1/2} \right\| \\
&\quad + \delta_0 + 2\delta_0 \qquad \text{by Equation 3.6} \\
&< \delta_1/8 + \delta_1/8 + 3\delta_0 \\
&< \varepsilon/2.
\end{aligned}
$$

Now suppose $1 < j \leq m$ and $j = l$. In this situation,

$$
\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_j y_{1,1}\phi\left(g\otimes e_{1,j}\right) - y_{j,j}\| \\
&\leq \|u_j y_{1,1} u_j^* u_j \phi(g\otimes e_{1,j}) - y_{j,j} u_j \phi(g\otimes e_{1,j})\| \\
&\quad + \|y_{j,j} u_j \phi(g\otimes e_{1,j}) - y_{j,j}\phi(g\otimes e_{j,j})\| \\
&\quad + \|y_{j,j}\phi(g\otimes e_{j,j}) - y_{j,j}\| \\
&\leq \delta_0 + \varepsilon/16 \text{ by Equations 3.10, 3.9, and 3.4} \\
&< \varepsilon/2.
\end{aligned}
$$

Now suppose $1 < j \leq m$ and $1 \leq l \leq m$ with $l \neq j$. Then,

$$
\begin{aligned}
\|u_j y_{1,l} - y_{j,l}\| &= \|u_j y_{1,1}\phi(g\otimes e_{1,l}) - y_{j,j}\phi(g\otimes e_{j,l})\| \\
&\leq \|u_j y_{1,1} u_j^* u_j \phi(g\otimes e_{1,l}) - u_j y_{1,1} u_j^* \phi(g\otimes e_{j,l})\| \\
&\quad + \|u_j y_{1,1} u_j^* \phi(g\otimes e_{j,l}) - y_{j,j}\phi(g\otimes e_{j,l})\| \\
&< \varepsilon/16 + \delta_0 \text{ by Equations 3.9 and 3.10} \\
&< \varepsilon/2.
\end{aligned}
$$

Finally, suppose $1 < j \leq m$ and $l = m+1$. Then

$$
\begin{aligned}
\|u_j y_{1,m+1} - y_{j,m+1}\| &\leq \|u_j\phi(f^{1/2}\otimes e_{1,1}) - \phi(f^{1/2}\otimes e_{j,1})\| \\
&< \varepsilon/2 \text{ by Equation 3.8.}
\end{aligned}
$$

Since we do not need to consider $j = 1$ because $u_j = 1$, this shows (5), and hence (6) hold.

For (3), we use Equation 3.5, namely that $\|y_{j,j} - a_j\| < \delta_0 < \varepsilon/192$. Combining this with Definition Lemma 2.7 we see that $\|y_{j,j}\| > 1 - \delta_6 - \delta_0 \geq 1 - \varepsilon/48 - \varepsilon/192 > 1/2$ by our assumption that $\varepsilon < 1$.

To check (7), we compute $\|y_{j,j} - a_j\|$ using Equation 3.10

$$\begin{aligned}\|y_{j,j} - a_j\| &\le \|y_{j,j} - u_j y_{1,1} u_j^*\| + \|u_j y_{1,1} u_j^* - u_j a_1 u_j^*\| + \|u_j a_1 u_j^* - a_j\| \\ &\le \delta_1/8 + \delta_5 + \delta_6 \\ &< \varepsilon.\end{aligned}$$

Next we check (8). Since $s_1 \in A$ and $u_{m+1}^* a_{m+1} u_{m+1} = \alpha_{m+1}^{-1}(a_{m+1}) \in A$, it is clear that $y_{1,1} \in A$.

Now we verify (9). For any $b \in F$ we have

$$\|y_{j,j}b - by_{j,j}\| \le \|y_{j,j}b - a_j b\| + \|a_j b - ba_j\| + \|ba_j - by_{j,j}\| < \varepsilon\|b\| + \delta_5 + \varepsilon\|b\| < 2\varepsilon\|b\| + \varepsilon.$$

For (4), we first recall that $0 \le a_g \le 1$ for all $g \in G$ by the definition of the projection free tracial Rokhlin property. Thus, $0 \le s_j u_{m+1}^* a_{m+1} u_{m+1} s_j^* \le s_j s_j^* = z_{j,j}$. The induction hypothesis that $0 \le z_{j,j} \le 1$ now gives us $0 \le y_{j,j}^2 \le 1$ which implies $0 \le y_{j,j} \le 1$ for $1 \le j \le m$. A similar argument shows that $0 \le y_{m+1,m+1} \le 1$.

Finally, we check the extra hypothesis for inducting, namely (13). Let $1 \le j, k \le m$ and $m + 1 < l \le n$. By the induction hypothesis, $0 = z_{j,k} a_l = \phi(t \otimes e_{j,k}) a_l$, and the same on the other side. Thus we also have $\phi(f^{1/2} \otimes e_{j,k}) a_l = 0$ and $\phi(g \otimes e_{j,k}) a_l = 0$. This implies $y_{m+1,j} a_l = 0$, and that $a_l y_{j,m+1} = 0$. We also have $a_l y_{j,j} = a_l y_{j,j} = 0$ and thus $y_{j,k} a_l = a_l y_{j,k} = 0$.

Since a_l is orthogonal to a_h for every other group element h, we also have $y_{m+1,m+1} a_l = a_l y_{m+1,m+1} = 0$. Similarly $y_{j,m+1} a_l = 0$ and $a_l y_{m+1,j} = 0$. This completes the induction step.

For the statement of the lemma, let $Y_{j,k}$ be given by the $y_{j,k}$ constructed when $m + 1 = n$, where $n = |G|$ and let a_g be as constructed in that same step. ∎

The following lemma is the projection free, finite group analog of Lemma 2.5 of [11]. It finds an isomorphic copy of matrices over a hereditary subalgebra of A as a large subalgebra of the crossed product. This is useful because we wish to show the entire crossed product has stable rank one and such a subalgebra has stable rank one.

Lemma 3.4. *Let A be an infinite dimensional stably finite simple unital C^*-algebra. Let G be a finite group; let $n = \mathrm{card}(G)$. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action with the projection free tracial Rokhlin property. Let $\iota : A \to C^*(G, A, \alpha)$ be the standard inclusion, write $B = C^*(G, A, \alpha)$, and let $u_g \in B$ be the standard unitary implementing α_g. Then for every finite set $F \subset B$, every $\varepsilon > 0$, and every natural number N, there exists a positive element $c^{(1)} \in B$, a subalgebra $D \subset \overline{c^{(1)} B c^{(1)}}$, a positive element $c_{1,1}^{(1)} \in A$, an isomorphism $\Phi : M_n \otimes \overline{c_{1,1}^{(1)} A c_{1,1}^{(1)}} \to D$ and elements $c_{g,h}^{(1)}$ for each g and h in G such that: With $\{e_{g,h}\}_{g,h \in G}$ being matrix units for M_n, we have*

(1) For any $d \in \overline{c_{1,1}^{(1)} A c_{1,1}^{(1)}}$ we have $\Phi(e_{1,1} \otimes d) = d$, and for any $s \in A$ with $\|s\| \le 1$ there are elements $d_g \in c_{1,1}^{(1)} A c_{1,1}^{(1)}$ such that $\Phi(e_{g,g} \otimes d_g) = c_{g,1}^{(1)} s c_{1,g}^{(1)}$ and $\mathrm{dist}(c_{g,1}^{(1)} s c_{1,g}^{(1)}, A) < \varepsilon$.

(2) $\|\Phi(e_{g,g} \otimes d) - u_g d u_g^*\| < \varepsilon\|d\|$ for all $d \in \overline{c_{1,1}^{(1)} A c_{1,1}^{(1)}}$.

(3) For all $x \in F$, there is a $y \in D$ such that $\|c^{(1)} x c^{(1)} - y\| < \varepsilon$ and $\|y\| \le \|x\|$.

(4) $\sum_{g \in G} \Phi(e_{g,g} \otimes c_{1,1}^{(1)}) = c^{(1)}$.

(5) $\|c^{(1)}x - xc^{(1)}\| < \varepsilon$ *for every* $x \in F$.
(6) $\tau(1 - c^{(1)}) < 1/N$ *for all* $\tau \in T(B)$.

Proof. Let F, ε, and N be given. Let $S \subset A$ be the finite set such that each element of F can be expressed as $\sum_{g\in G} b_g u_g$ with coefficients b_g in S.

Without loss of generality, $\|x\| \leq 1$ for all $x \in F$ and $\|y\| \leq 1$ for all $y \in S$. We can always rescale to achieve this.

First we observe that by the same argument used in the proof of Lemma 2.5 of [11] we do not need to prove the norm condition in (3) above.

Let $0 < \varepsilon_0 < \min\{\varepsilon/(40n^2), \varepsilon/(12)\}$. Define continuous functions f_0 and f_1 on $[0, 1]$ as follows:

$f_0(0) = 0$,
$f_0(t) = 1$ for t in $[1 - \varepsilon_0, 1]$, and
f_0 is linear on $(0, 1 - \varepsilon_0)$.
$f_1(t) = 0$ for t in $[0, 1 - \varepsilon_0]$,
$f_1(t) = 1$ for t in $[1 - \varepsilon_0/2, 1]$, and
f_1 is linear on $(1 - \varepsilon_0, 1 - \varepsilon_0/2)$.

Let $0 < \varepsilon_1 < \min\{\varepsilon/(8n^2), \varepsilon/(12)\}$. Apply Lemma 2.12 to f_1 with ε_1 in place of ε to get δ_1. Apply Lemma 2.13 to f_1 with ε_1 in place of ε to get δ_2.

Let

$$0 < \varepsilon_2 < \min\left\{\frac{\varepsilon}{28n^2}, \frac{\varepsilon}{12}, \frac{\delta_1}{4}, \frac{\delta_2}{5n^2}, \frac{1}{2nN}\right\}.$$

Let δ_3 be the value of δ given by applying Lemma 2.14 with $\min\{\frac{\varepsilon_0}{2}, n\varepsilon_2 + \frac{2}{N}\}$ in place of ε. We also require $\delta_3 < \frac{1}{N}$.

Let $\{e_{g,h}\}$ for $g, h \in G$ be a system of matrix units for M_n. Let t represent the function $f(t) = t$. Notice that $\{t \otimes e_{g,h}\}_{g,h\in G}$ generate CM_n.

Apply Lemma 3.3 with S in place of F and with ε_2 in place of ε. This provides us with $\delta > 0$, $a_g \in A$ for $g \in G$ and $Y_{g,h} \in B$ for $g, h \in G$ satisfying the conclusions of that lemma. Thus we can define a homomorphism, $\varphi_0 : CM_n \to B$ given by $\varphi_0(t \otimes e_{g,h}) = Y_{g,h}$. We also require $\delta < \frac{1}{2N}$.

Let $c^{(0)}_{g,h} = \varphi_0(f_0 \otimes e_{g,h})$. Similarly define $c^{(1)}_{g,h} = \varphi_0(f_1 \otimes e_{g,h})$. Also set $c^{(0)} = \sum_{g\in G} c^{(0)}_{g,g}$ and similarly $c^{(1)} = \sum_{g\in G} c^{(1)}_{g,g}$. Notice that since $Y_{1,1} \in A$, we also have $c^{(0)}_{1,1} \in A$ and $c^{(1)}_{1,1} \in A$.

Notice $c^{(0)}_{g_1,h_1} c^{(1)}_{g_2,h_2} = \delta_{h_1,g_2} c^{(1)}_{g_1,h_2}$. Similarly, $c^{(1)}_{g_2,h_2} c^{(0)}_{g_1,h_1} = \delta_{h_2,g_1} c^{(1)}_{g_2,h_1}$.

Define a function $\Phi : M_n(\overline{c^{(1)}_{1,1} A c^{(1)}_{1,1}}) \to B$ by $\Phi((x_{g,h})) = \sum_{g,h} c^{(0)}_{g,1} x_{g,h} c^{(0)}_{1,h}$ for $x_{g,h} \in \overline{c^{(1)}_{1,1} A c^{(1)}_{1,1}}$. Set $D = \mathrm{Im}(\Phi)$.

Next we check that Φ is a homomorphism. It is easy to check that Φ is additive and is star preserving. We will check that it is multiplicative.

Let $x = (x_{g,h})$ and $y = (y_{g,h})$. Note that $(xy)_{g,h} = \sum_{k\in G} x_{g,k}y_{k,h}$. Then, using the facts that $x_{g,h}$ and $y_{k,l}$ are in $\overline{c_{1,1}^{(1)}Ac_{1,1}^{(1)}}$ and that $c_{1,1}^{(1)}c_{1,1}^{(0)} = c_{1,1}^{(1)}$, we get:

$$\begin{aligned}\Phi(x)\Phi(y) &= \left[\sum_{g,h\in G} c_{g,1}^{(0)}x_{g,h}c_{1,h}^{(0)}\right]\left[\sum_{k,l\in G} c_{k,1}^{(0)}x_{k,l}c_{1,l}^{(0)}\right]\\ &= \sum_{g,h,l\in G} c_{g,1}^{(0)}x_{g,h}y_{h,l}c_{1,l}^{(0)}\\ &= \sum_{g,l\in G} c_{g,1}^{(0)}\left(\sum_{h\in G} x_{g,h}y_{h,l}\right)c_{1,l}^{(0)}\\ &= \Phi(xy).\end{aligned}$$

Furthermore, Φ is injective since $\overline{c_{1,1}^{(1)}Ac_{1,1}^{(1)}}$ is simple by Theorem 3.2.8 of [10] and $\Phi(c_{1,1}^{(1)}) = c_{1,1}^{(0)}c_{1,1}^{(1)}c_{1,1}^{(0)} = c_{1,1}^{(1)} \neq 0$.

Next we make some norm estimates to be used later on. Note we have $\varphi_0(f_1 \otimes e_{g,g}) = f_1(\varphi_0(t \otimes e_{g,g}))$ by Lemma 3.1. Also note that $\|c_{h,k}^{(0)} - Y_{h,k}\| \leq \|f_0 - t\| < \varepsilon_0$.

Next we estimate the affect of conjugating $c_{h,h}^{(1)}$ by u_g. Since

$$\|u_gY_{h,h}u_g^* - Y_{gh,gh}\| < 2\varepsilon_2 < \delta_1,$$

by the choice of δ_1. Using Lemma 2.12, we have

$$\begin{aligned}\|u_gc_{h,h}^{(1)}u_g^* - c_{gh,gh}^{(1)}\| &= \|f_1(u_g(\varphi_0(t\otimes e_{h,h}))u_g^*) - f_1(\varphi_0(t\otimes e_{gh,gh}))\|\\ &< \varepsilon_1.\end{aligned}\tag{3.11}$$

Now we compute,

$$\begin{aligned}\|u_gc_{h,k}^{(0)} - c_{gh,k}^{(0)}\| &\leq \|u_gc_{h,k}^{(0)} - u_gY_{h,k}\| + \|u_gY_{h,k} - Y_{gh,k}\| + \|Y_{gh,k} - c_{gh,k}\|\\ &\leq 2\varepsilon_0 + \varepsilon_2.\end{aligned}\tag{3.12}$$

Next, using Equations 3.11 and 3.12 for the second inequality, we compute the similar quantity using $c_{h,k}^{(1)}$:

$$\begin{aligned}\|u_gc_{h,k}^{(1)} - c_{gh,k}^{(1)}\| &\leq \|u_gc_{h,h}^{(1)}u_g^*u_gc_{h,k}^{(0)} - c_{gh,gh}^{(1)}u_gc_{h,k}^{(0)}\| + \|c_{gh,gh}^{(1)}u_gc_{h,k}^{(0)} - c_{gh,gh}^{(1)}c_{gh,k}^{(0)}\|\\ &\leq \varepsilon_1 + 2\varepsilon_0 + \varepsilon_2.\end{aligned}\tag{3.13}$$

Let $s \in S$ and recall that we have normalized so that $\|s\| \leq 1$ for all $s \in S$. We have

$$\begin{aligned}\|[Y_{g,g}, s]\| &\leq \|Y_{g,g}s - a_gs\| + \|a_gs - sa_g\| + \|sa_g - sY_{g,g}\|\\ &\leq 2\varepsilon_2 + \delta_3\\ &\leq \delta_2.\end{aligned}$$

Using the preceding estimate and the choice of δ_2 using Lemma 3.1 we now get

$$\begin{aligned}\left\|\left[c_{g,g}^{(1)}, s\right]\right\| &\leq \|[\,f_1(\varphi_0(t\otimes e_{g,g})), s\,]\|\\ &< \varepsilon_1.\end{aligned}\tag{3.14}$$

Let $y \in B$ and $g, h, k, l \in G$. Then we observe

$$\begin{aligned} \left\| c^{(0)}_{g,h} y c^{(0)}_{k,l} - Y_{g,h} y Y_{k,l} \right\| &\leq 2 \|y\| \|t - f_0\| \\ &\leq 2 \|y\| \varepsilon_0. \end{aligned} \tag{3.15}$$

Let $y \in B$. Then by Equation 3.12 we have

$$\begin{aligned} \left\| c^{(0)}_{h,1} y c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y u_h^* c^{(0)}_{h,h} u_g \right\| &\leq \left\| c^{(0)}_{h,1} y c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y c^{(0)}_{1,g^{-1}h} \right\| \\ &\quad + \left\| c^{(0)}_{h,h} u_h y c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y u_h^* c^{(0)}_{h,h} u_g \right\| \\ &< 3 \|y\| (2\varepsilon_0 + \varepsilon_2). \end{aligned} \tag{3.16}$$

Now let $\{y_{g,h}\} \subset B$ for $g, h \in G$. Then,

$$\begin{aligned} &\left\| \sum_{g,h \in G} c^{(0)}_{h,1} y_{h,g^{-1}h} c^{(0)}_{1,g^{-1}h} - \sum_{g,h \in G} c^{(0)}_{h,h} u_h y_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g \right\| \\ &\leq \sum_{g,h \in G} \left\| c^{(0)}_{h,1} y_{h,g^{-1}h} c^{(0)}_{1,g^{-1}h} - c^{(0)}_{h,h} u_h y_{h,g^{-1}h} u_h^* c^{(0)}_{h,h} u_g \right\| \\ &\leq 3n^2 \max_{g,h \in G} \left\| y_{h,g^{-1}h} \right\| (2\varepsilon_0 + \varepsilon_2). \end{aligned} \tag{3.17}$$

Now given $x \in F$, we can write $x = \sum_{g \in G} x_g u_g$ with $x_g \in S$.

Set $z_{h,g} = u_h^* (c^{(1)}_{h,h} (x_{hg^{-1}}) c^{(1)}_{h,h}) u_h$ and $y_{h,g} = c^{(1)}_{1,1} \alpha_{h^{-1}} (x_{hg^{-1}}) c^{(1)}_{1,1}$. Note that $y_{h,g} \in \overline{c^{(1)}_{1,1} A c^{(1)}_{1,1}}$. Then, using Equation 3.11 and the fact that $\|x_g\| \leq 1$ we compute:

$$\begin{aligned} \|z_{h,g^{-1}h} - y_{h,g^{-1}h}\| &= \|u_h^* c^{(1)}_{h,h} x_g c^{(1)}_{h,h} u_h - c^{(1)}_{1,1} \alpha_{h^{-1}}(x_g) c^{(1)}_{1,1}\| \\ &\leq 2\varepsilon_1. \end{aligned} \tag{3.18}$$

Next we estimate the effect of $c^{(1)}$ on x. In the second step we used Equation 3.11. For the second to last step we used Equation 3.14 and the fact that $c^{(1)}_{h,h} c^{(1)}_{g,g} = 0$ unless $g = h$. For the last step we used the fact that $\|x_g\| \leq 1$, since $x_g \in S$ to compute:

$$\begin{aligned}
\left\| c^{(1)}xc^{(1)} - \sum_{g,h\in G} c^{(0)}_{h,h}c^{(1)}_{h,h}x_g c^{(1)}_{h,h}c^{(0)}_{h,h}u_g \right\| &\le \left\| \sum_{h,g,k\in G} c^{(1)}_{h,h}x_g u_g c^{(1)}_{k,k} - \sum_{g,h,k\in G} c^{(1)}_{h,h}x_g c^{(1)}_{gk,gk}u_g \right\| \\
&\quad + \left\| \sum_{g,h,k\in G} c^{(1)}_{h,h}x_g c^{(1)}_{gk,gk}u_g - \sum_{g,h} c^{(1)}_{h,h}x_g c^{(1)}_{h,h}u_g \right\| \\
&\le n^2 \max_{g\in G} \|x_g\|\,\varepsilon_1 \\
&\quad + \left\| \sum_{g,h,k\in G} c^{(1)}_{h,h}x_g c^{(1)}_{gk,gk}u_g - \sum_{g,h,k\in G} c^{(1)}_{h,h}c^{(1)}_{gk,gk}x_g u_g \right\| \\
&\quad + \left\| \sum_{g,h,k\in G} c^{(1)}_{h,h}c^{(1)}_{gk,gk}x_g u_g - \sum_{h,g\in G} (c^{(1)}_{h,h})^2 x_g u_g \right\| \\
&\quad + \left\| \sum_{h,g\in G} (c^{(1)}_{h,h})^2 x_g u_g - \sum_{h,g\in G} c^{(1)}_{h,h}x_g c^{(1)}_{h,h}u_g \right\| \\
&\le n^2 \max_{g\in G} \|x_g\|\,\varepsilon_1 + n^2 \max_{g\in G} \|x_g\|\,\varepsilon_1 + 0 + n^2 \max_{g\in G} \|x_g\|\,\varepsilon_1 \\
&\le 3n^2\varepsilon_1. \qquad (3.19)
\end{aligned}$$

We are now in a position to prove part (3) of the statement. Note that $\Phi((y_{h,g^{-1}h})) = \sum_{g,h\in G} c^{(0)}_{h,1}y_{h,g^{-1}h}c^{(0)}_{1,g^{-1}h} \in D$ and so

$$\begin{aligned}
\left\| \Phi((y_{h,g^{-1}h})) - c^{(1)}xc^{(1)} \right\| &= \left\| \sum_{g,h\in G} c^{(0)}_{h,1}y_{h,g^{-1}h}c^{(0)}_{1,g^{-1}h} - c^{(1)}xc^{(1)} \right\| \\
&\le \left\| \sum_{g,h\in G} c^{(0)}_{h,1}y_{h,g^{-1}h}c^{(0)}_{1,g^{-1}h} - \sum_{g,h\in G} c^{(0)}_{h,h}u_h y_{h,g^{-1}h}u_h^* c^{(0)}_{h,h}u_g \right\| \\
&\quad + \left\| \sum_{g,h\in G} c^{(0)}_{h,h}u_h y_{h,g^{-1}h}u_h^* c^{(0)}_{h,h}u_g - \sum_{g,h\in G} c^{(0)}_{h,h}u_h z_{h,g^{-1}h}u_h^* c^{(0)}_{h,h}u_g \right\| \\
&\quad + \left\| \sum_{g,h} c^{(0)}_{h,h}c^{(1)}_{h,h}x_g c^{(1)}_{h,h}c^{(0)}_{h,h}u_g - c^{(1)}xc^{(1)} \right\| \\
&\le 3n^2 \max_{g,h\in G} \|y_{h,g^{-1}h}\|(2\varepsilon_0 + \varepsilon_2) \\
&\quad + n^2 2\varepsilon_0 \\
&\quad + 3n^2\varepsilon_1 \ \text{ by Equations 3.17, 3.18, and 3.19} \\
&< \varepsilon.
\end{aligned}$$

This proves part (3) of the statement with y taken to be $\Phi((y_{h,g^{-1}h}))$.

For part (1) of the conclusion, suppose $d \in \overline{c_{1,1}^{(1)}Ac_{1,1}^{(1)}}$. Then

$$\Phi(e_{1,1}\otimes d) = c_{1,1}^{(0)}dc_{1,1}^{(0)} = \lim_{n\to\infty} c_{1,1}^{(0)}(c_{1,1}^{(1)})^{1/n}d(c_{1,1}^{(1)})^{1/n}c_{1,1}^{(0)} = \lim_{n\to\infty}(c_{1,1}^{(1)})^{1/n}d(c_{1,1}^{(1)})^{1/n} = d.$$

This is the first half of (1).

For the second part of (1), let $s \in S$. Recall that we have normalized so that $\|s\| \leq 1$. Let $d = c_{1,1}^{(1)}sc_{1,1}^{(1)} \in c_{1,1}^{(1)}Ac_{1,1}^{(1)}$. Then

$$\Phi(e_{g,g}\otimes d) = c_{g,1}^{(0)}dc_{1,g}^{(0)} = c_{g,1}^{(0)}c_{1,1}^{(1)}sc_{1,1}^{(1)}c_{1,g}^{(0)} = c_{g,1}^{(1)}sc_{1,g}^{(1)}.$$

Furthermore,

$$\begin{aligned}
\operatorname{dist}(c_{g,1}^{(1)}sc_{1,g}^{(1)}, A) &\leq \|c_{g,1}^{(1)}sc_{1,g}^{(1)} - u_g c_{1,1}^{(1)}sc_{1,1}^{(1)}u_g^*\| \\
&\leq \|c_{g,1}^{(1)} - u_g c_{1,1}^{(1)}\| + \|c_{1,g}^{(1)} - c_{1,1}u_g^*\| \\
&\leq 2(\varepsilon_1 + 2\varepsilon_0 + \varepsilon_2) \quad \text{by Equation 3.13} \\
&< \varepsilon.
\end{aligned}$$

This completes (1).

To prove (2), let $d \in \overline{c_{1,1}^{(1)}Ac_{1,1}^{(1)}}$. Using the fact that $d = \lim_{m\to\infty}(c_{1,1}^{(1)})^{1/m}d(c_{1,1}^{(1)})^{1/m}$, we compute:

$$\begin{aligned}
\|u_g d u_g^* - \Phi(e_{g,g}\otimes d)\| &= \left\|\lim_{m\to\infty} u_g c_{1,1}^{(0)}(c_{1,1}^{(1)})^{1/m}d(c_{1,1}^{(1)})^{1/m}c_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,g}^{(0)}\right\| \\
&\leq \left\|u_g c_{1,1}^{(0)}dc_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,1}^{(0)}u_g^*\right\| + \left\|c_{g,1}^{(0)}dc_{1,1}^{(0)}u_g^* - c_{g,1}^{(0)}dc_{1,g}^{(0)}\right\| \\
&\leq \left\|u_g c_{1,1}^{(0)} - c_{g,1}^{(0)}\right\|\|d\| + \left\|c_{1,1}^{(0)}u_g^* - c_{1,g}^{(0)}\right\|\|d\| \\
&\leq 2(2\varepsilon_0 + \varepsilon_2)\|d\| \ \text{ using Equation 3.12} \\
&< \varepsilon\|d\|.
\end{aligned}$$

This is condition (2) of the lemma.

For (4) we compute

$$\sum_{g\in G}\Phi(e_{g,g}\otimes c_{1,1}^{(1)}) = \sum_{g\in G} c_{g,1}^{(0)}c_{1,1}^{(1)}c_{1,g}^{(0)} = \sum_{g\in G} c_{g,g}^{(1)} = c^{(1)}.$$

For (5) we begin by computing for any $x = \sum_{h\in G} x_h u_h \in F$ how close x and $\sum_{g\in G} Y_{g,g}$ are to commuting. We have

$$\begin{aligned}
&\left\|\left(\sum_{g\in G} Y_{g,g}\right)\left(\sum_{h\in G} x_h u_h\right) - \left(\sum_{h\in G} x_h u_h\right)\left(\sum_{g\in G} Y_{g,g}\right)\right\| \\
&\leq \sum_{g,h\in G}\|Y_{g,g}x_h u_h - x_h Y_{g,g}u_h\| + \sum_{g,h\in G}\left\|x_h u_h u_h^* Y_{g,g}u_h - x_h u_h Y_{h^{-1}g,h^{-1}g}\right\| \\
&\leq 3n^2\varepsilon_2 + 2n^2\varepsilon_2 \\
&< \delta_2.
\end{aligned}$$

By the choice of δ_2, this implies $\|f_1(\sum_{g\in G} Y_{g,g})x - xf_1(\sum_{g\in G} Y_{g,g})\| < \varepsilon_1 < \varepsilon$. But, by Lemma 2.11, we have $\varepsilon > \|\sum_{g\in G} f_1(Y_{g,g})x - x\sum_{g\in G} f_1(Y_{g,g})\| = \|c^{(1)}x - xc^{(1)}\|$ which is (5).

Finally, we will show that (6) holds. We wish to show that $\tau(1 - c^{(1)}) < \frac{1}{N}$ for all $\tau \in T(B)$. However, in light of Lemma 2.15, it suffices to prove the statement

for all $\tau \in T(A)$. Now, since $\|a_g - Y_{g,g}\| < \varepsilon_2$, we have $\|\sum_{g\in G} a_g - \sum_{g\in G} Y_{g,g}\| < n\varepsilon_2$. Therefore, $\tau(\sum_{g\in G} a_g) < n\varepsilon_2 + \tau(\sum_{g\in G} Y_{g,g})$. By the choice of δ using Lemma 3.3 we have $\tau(1 - \sum_{g\in G} a_g) \leq \frac{1}{2N}$.

Combining these facts we have

$$\frac{1}{2N} > 1 - \tau\left(\sum_{g\in G} a_g\right) > 1 - \tau\left(\sum_{g\in G} Y_{g,g}\right) - n\varepsilon_2.$$

This implies

$$\frac{1}{2N} + n\varepsilon_2 > \tau(1 - \sum_{g\in G} Y_{g,g}).$$

Therefore, by the choice of δ_3,

$$\frac{1}{N} > \delta_3 > \tau\left(1 - f_1\left(\sum_{g\in G} Y_{g,g}\right)\right) = \tau\left(1 - c^{(1)}\right)$$

which is (6). ∎

The following lemma is the analog for positive elements of Lemma 3.2 of [11].

Lemma 3.5. *Let A be a C^*-algebra, let $x, y \in A_+$, let τ be a tracial state on A. Let $g : [0,1] \to \mathbb{R}$ be a continuous function. Then $\tau(g(y^{1/2}xy^{1/2})) = \tau(g(x^{1/2}yx^{1/2})$.*

Proof. We first verify the statement for $g(t) = t^n + c$ for any natural number n and constant c. Note that

$$\begin{aligned}\tau((y^{1/2}xy^{1/2})^n + c) &= \tau(y^{1/2}(xy)^{n-1}x^{1/2}(x^{1/2}y^{1/2})) + c \\ &= \tau((x^{1/2}y^{1/2})y^{1/2}(xy)^{n-1}x^{1/2}) + c \\ &= \tau((x^{1/2}yx^{1/2})^n + c)\end{aligned}$$

Thus the lemma holds for any polynomial and so, by the continuity of functional calculus, for any continuous function. ∎

Lemma 3.9 is an analog of Lemma 3.3 of [11] for positive elements instead of projections. The next few lemmas are used to prove Lemma 3.9.

Lemma 3.6. *Let $g : [0,1] \to [0,1]$ be a continuous function with $g(1) = 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and x, y are positive elements of A with $\|x\| \leq 1$ and $\|y\| \leq 1$ such that $\tau(x) > 1 - \delta$ and $\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta$, then $\tau(g(yxy)) > \tau(y^2) - \varepsilon$.*

Proof. Choose $\delta_0 \in (0,1)$ such that $g(t) > 1 - \varepsilon/2$ for all $t \in [1-\delta_0, 1]$. Choose δ so that $\delta < \frac{\varepsilon\delta_0}{4}$. Let A, τ, x, and y be as in the hypotheses.

We first estimate $\tau(yxy)$. We have $\tau(yxy) + \tau(y(1-x)y) = \tau(y^2)$. By the condition on x and since $y \leq 1$ implies $(1-x)^{1/2}y^2(1-x)^{1/2} \leq 1 - x$, we also have

$$\tau(y(1-x)y) = \tau\left((1-x)^{1/2}y^2(1-x)^{1/2}\right) \leq \tau(1-x) < \delta.$$

Combining these two observations yields

$$\tau(yxy) = \tau(y^2) - \tau(y(1-x)y) > \tau(y^2) - \delta. \tag{3.20}$$

Now restrict τ to $\overline{yAy}$. Call the restriction $\hat{\tau}$. Extend $\hat{\tau}$ to a trace $\overline{\tau}$ on $\overline{yAy} + \mathbb{C}1_A$ by $\overline{\tau}(1_A) = \|\hat{\tau}\|$. This implies that

$$\|\overline{\tau}\| = \|\hat{\tau}\|. \tag{3.21}$$

Let μ be the measure on $X = \mathrm{sp}(yxy)$ corresponding to the functional on $C(X)$ defined by $h \mapsto \overline{\tau}(h(yxy))$ with the functional calculus evaluated in $\overline{yAy} + \mathbb{C}1_A$. That is $\int_X h d\mu = \overline{\tau}(h(yxy))$.

The total mass of μ is $\|\hat{\tau}\|$.

Let $E = [1 - \delta_0, 1]$. We compute

$$\begin{aligned}
\tau(y^2) - \delta < \tau(yxy) & \qquad \text{by } 3.20 \\
= \int_{[0,1]} t d\mu(t) & \qquad \text{by the definition of } \mu \\
\leq (1 - \delta_0)(\|\hat{\tau}\| - \mu(E)) + \mu(E) & \\
= (1 - \delta_0)\|\hat{\tau}\| + \delta_0 \mu(E) & \\
< (1 - \delta_0)(\tau(y^2) + \delta) + \delta_0 \mu(E) & \qquad \text{by hypothesis.}
\end{aligned}$$

This implies

$$\begin{aligned}
\tau(y^2) - \delta - (1 - \delta_0)(\tau(y^2) + \delta) &< \delta_0 \mu(E) \\
-2\delta + \delta_0 \tau(y^2) + \delta_0 \delta &< \delta_0 \mu(E) \\
\tau(y^2) - \frac{2\delta}{\delta_0} + \delta &< \mu(E) \\
\tau(y^2) - \frac{2\delta}{\delta_0} &< \mu(E) \\
\tau(y^2) - \frac{\varepsilon}{2} &< \mu(E).
\end{aligned}$$

Since $g(t) > 1 - \varepsilon/2$ for $t \in E$, by using $\tau(y^2) \leq 1$ for the last inequality, we now get

$$\begin{aligned}
\tau(g(yxy)) &= \int_{[0,1]} g(t) d\mu(t) \\
&\geq (1 - \varepsilon/2)\mu(E) \\
&\geq \tau(y^2) - \varepsilon/2 - \tau(y^2)\varepsilon/2 + \varepsilon/4 \\
&\geq \tau(y^2) - \varepsilon.
\end{aligned}$$

This completes the proof. ∎

Lemma 3.7. *Given any $\delta > 0$, there exists an $\eta > 0$ such that whenever A is a unital C^*-algebra and $y \in A$ is a positive element of norm less than or equal to 1 and $\tau \in T(A)$, with $\tau(y) > \|\tau|_{\overline{yAy}}\| - \eta$, then $\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta$.*

Proof. Apply Lemma 3.6 with ε replaced by $\delta/2$ and with $g(t) = t^2$. Let η be the resulting value of δ. Without loss of generality, $\eta < \delta/2$. Let $y \in A$ be a positive element with $\|y\| \leq 1$ be such that $\tau((y^{1/2})^2) = \tau(y) > \|\tau|_{\overline{yAy}}\| - \eta = \|\tau|_{\overline{y^{1/2}Ay^{1/2}}}\| - \eta$. Then by the choice of η using Lemma 3.6 and letting $x = 1$ and using $y^{1/2}$ in place of y yields

$$\begin{aligned}\tau\left(g\left(y^{1/2}xy^{1/2}\right)\right) &> \tau((y^{1/2})^2) - \delta/2 \\ \tau(y^2) &> \tau(y) - \delta/2 \\ \tau(y^2) &> \|\tau|_{\overline{yAy}}\| - \delta.\end{aligned}$$

This completes the proof. ∎

Lemma 3.8. *Let $g : [0,1] \to [0,1]$ be a continuous function with $g(1) = 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and x, y are positive elements of A with norm less than or equal to 1 satisfying $\tau(x) > 1-\delta$ and $\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta$, we have $\tau(g(yxy)) > \tau(y^2) - \varepsilon$.*

Proof. Let δ_1 be the δ obtained by applying Lemma 3.6 with ε and g as given. Let δ_2 be the η obtained by applying Lemma 3.7 with δ replaced by δ_1. Let $\delta_3 = \min\{\delta_1, \delta_2\}$. If $\tau(x) > 1 - \delta_3$, then $\tau(x) > 1 - \delta_1$, so the condition on x is satisfied in Lemma 3.6. If

$$\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta_3 > \|\tau|_{\overline{yAy}}\| - \delta_2,$$

then

$$\tau(y^2) > \|\tau|_{\overline{yAy}}\| - \delta_1$$

by the choice of δ_2 using Lemma 3.7. Thus the condition on y in Lemma 3.6 is satisfied and therefore $\tau(g(yxy)) > \tau(y^2) - \varepsilon$. ∎

Lemma 3.9. *Let $g : [0,1] \to [0,1]$ be a continuous function with $g(1) = 1$. For every $\varepsilon > 0$, there exists $\delta > 0$ such that whenever A is a unital C^*-algebra, τ is a tracial state on A, and x, y are positive elements of A with norm less than or equal to 1 satisfying $\tau(x) > 1 - \delta$ and $\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta$, then $\tau(g(xyx)) > \tau(y) - \varepsilon$.*

Proof. Apply Lemma 3.6 with g and ε as given to get $\delta_1 > 0$. Now apply Lemma 3.8 with $g(t) = t^2$ and δ_1 in place of ε to get δ_2. Let δ_3 be the δ obtained from applying Lemma 3.8 with g as given and ε as given. We may assume $\delta_3 < \delta_2 < \delta_1$. The number δ_3 is the desired δ. By the choice of δ_2 using Lemma 3.8 with 1 in place of y for any x satisfying $\tau(x) > 1 - \delta_2$ we have
$\tau(g(yxy)) = \tau(g(x)) = \tau(x^2) > 1 - \delta_1$. So now if y is such that
$\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta_2 > \|\tau|_{\overline{yAy}}\| - \delta_3$, by the choice of δ_3 using Lemma 3.6 with $y^{1/2}$ in place of y we have $\tau\left(g\left(y^{1/2}x^2y^{1/2}\right)\right) > \tau\left(\left(y^{1/2}\right)^2\right) - \varepsilon$. But by Lemma 3.5, $\tau\left(g\left(y^{1/2}x^2y^{1/2}\right)\right) = \tau(g(xyx))$. Thus $\tau(g(xyx)) > \tau(y) - \varepsilon$. Additionally, since $\tau(x) > 1 - \delta_2 > 1 - \delta_3$ and $\tau(y) > \|\tau|_{\overline{yAy}}\| - \delta_3$, we have $\tau(g(yxy)) > \tau(y^2) - \varepsilon$. ∎

The following lemma is an analog for positive elements of Lemma 5.1 of [11].

Lemma 3.10. *Let $\delta > 0$. There exists a continuous function $g : [0,1] \to [0,1]$ such that $g(0) = 0$, $g(1) = 1$, and whenever A is a C^*-algebra and $a \in A$ is positive with $\|a\| \leq 1$, then there is a positive element $b \in \overline{aAa}$ with $\|b\| \leq 1$ such that $\|bg(a) - g(a)\| < \delta$ and $\|ab - b\| < \delta$.*

Proof. Choose t_0 and t_1 with $1 - \delta < t_0 < t_1 < 1$ and let $g : [0,1] \to [0,1]$ be a continuous function which vanishes on $[0, t_1]$ and such that $g(1) = 1$. Let A be a

C^*-algebra, and let $a \in A$ be positive with $\|a\| \leq 1$. Let $h : [0,1] \to [0,1]$ be a continuous function which vanishes on $[0, t_0]$ such that $h(t) = 1$ for $t \in [t_1, 1]$.

For n sufficiently large, $\|g(a)^{1/n} g(a) - g(a)\| < \delta$. So let $b = g(a)^{1/n}$. Note that since $g(a)^{1/n}$ is positive, $(\|g(a)^{1/n}\|)^n = \|g(a)\| = 1$, which implies that $\|(g(a))^{1/n}\| = 1$. From $hg = g$ we have $h(a)g(a) = g(a)$ and so $h(a)b = b$. Also $\|ah(a) - h(a)\| < \delta$ because $|t-1| \leq 1 - t_0 < \delta$ whenever $h(t) \neq 0$. Accordingly, we have

$$\|ab - b\| = \|ah(a)b - h(a)b\| \leq \|ah(a) - h(a)\|\|b\| < \delta,$$

which completes the proof. ∎

The next lemma is used repeatedly, but often implicitly in the proof of Lemma 3.12.

Lemma 3.11. *If y and z are orthogonal positive elements of a C^*-algebra A and $w \in \overline{Ay}$ and $x \in \overline{zA}$, then $wx = 0$ as well.*

Proof. We have $wx = \lim_{n\to\infty} \lim_{m\to\infty} wy^{1/n} z^{1/m} x = w \cdot 0 \cdot x = 0$. ∎

The following lemma is used in the proof of the main theorem, Theorem 3.17, to simulate a decomposition of the identity into orthogonal projections.

Lemma 3.12. *Let $\varepsilon > 0$. Suppose $b_1, b_2, b_3, c_1, c_2, c_3$ are positive elements of a stably finite unital C^*-algebra A, and let $a \in A$. Suppose:*

- $b_1 + b_2 + b_3 = 1$,
- $c_1 + c_2 + c_3 = 1$,
- $C^*(b_1, b_2, b_3, c_1, c_2, c_3)$ *is commutative,*
- $b_1 c_1 = c_1$,
- $b_3 c_3 = c_3$,
- $b_2 c_2 = b_2$,
- $c_1 b_2 = c_3 b_2 = 0$,
- $b_1 b_3 = 0$,
- $\overline{c_2 A c_2}$ *have stable rank one, and*
- $b_1 a = a b_3 = 0$.

Then there exists an element $a_1 \in A$ such that a_1 is invertible and $\|a - a_1\| < \varepsilon$.

Proof. Write $1 = c_1 + (b_1 - c_1) + b_2 + (b_3 - c_3) + c_3$. We wish to use this decomposition of the identity to decompose a. Therefore, make the following definitions:

$a_{3,1} = b_2 a c_1$.
$a_{3,2} = b_2 a (b_1 - c_1)$.
$a_{3,3} = b_2 a b_2$.
$a_{4,1} = (b_3 - c_3) a c_1$.
$a_{4,2} = (b_3 - c_3) a (b_1 - c_1)$.
$a_{4,3} = (b_3 - c_3) a b_2$.
$a_{5,1} = c_3 a c_1$.
$a_{5,2} = c_3 a (b_1 - c_1)$.
$a_{5,3} = c_3 a b_2$.

Notice that $\sum_{i=3}^{5} \sum_{j=1}^{3} a_{i,j} = a$.

Let

$$0 < \delta < \min\left\{ \frac{\varepsilon}{6\|a_{3,1} + a_{4,1} + a_{5,1}\|}, \sqrt{\frac{\varepsilon}{6}}, \frac{\varepsilon}{3} \right\}.$$

Since $a_{3,3} \in \overline{b_2 A b_2}$, there is an invertible element $t_0 \in \overline{b_2 A b_2} + \mathbb{C}1_A$ with

$$\|t_0 - a_{3,3}\| < \delta. \tag{3.22}$$

Write $t_0 = t_1 + \lambda_1 1_A$ with $t_1 \in \overline{b_2 A b_2}$ and $\lambda_1 \in \mathbb{C}$. We can also express t_0^{-1} as $t_2 + \lambda_1^{-1} 1_A$ with $t_2 \in \overline{b_2 A b_2}$.

Next we show that

$$((a_{3,1} + a_{4,1} + a_{5,1}) t_0^{-1})^2 = 0. \tag{3.23}$$

We note that

$$(a_{3,1} + a_{4,1} + a_{5,1}) t_0 = (a_{3,1} + a_{4,1} + a_{5,1})(t_1 + \lambda_1) = (a_{3,1} + a_{4,1} + a_{5,1})\lambda_1.$$

Therefore,

$$(a_{3,1} + a_{4,1} + a_{5,1}) = (a_{3,1} + a_{4,1} + a_{5,1}) t_0 t_0^{-1} = (a_{3,1} + a_{4,1} + a_{5,1})\lambda_1 t_0^{-1}.$$

This implies

$$((a_{3,1} + a_{4,1} + a_{5,1}) t_0^{-1})^2 = (a_{3,1} + a_{4,1} + a_{5,1})^2 \lambda_1^{-2} = 0.$$

Now we compute

$$\begin{aligned} & (a_{3,1} + a_{4,1} + a_{5,1} + t_0)\, t_0^{-1} \left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right) \\ & = (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1} - \left((a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right)^2 \\ & \quad + 1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1} \\ & = 1. \end{aligned} \tag{3.24}$$

Because A is stably finite this is enough to show that $a_{3,1} + a_{4,1} + a_{5,1} + t_0$ and
$t_0^{-1} \left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right)$ are mutual inverses.

Next we multiply

$$\begin{aligned} & t_0^{-1} \left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right) \\ & \quad \cdot (a_{3,1} + a_{3,2} + t_0 + a_{4,1} + a_{4,2} + a_{4,3} + a_{5,1} + a_{5,2} + a_{5,3}) \\ & = 1 + t_0^{-1} \left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right) \\ & \quad \cdot (a_{3,2} + a_{4,2} + a_{4,3} + a_{5,2} + a_{5,3}). \end{aligned} \tag{3.25}$$

Using our expression for t_0^{-1} we can compute

$$\left((a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right) b_3 = (a_{3,1} + a_{4,1} + a_{5,1}) \left(t_2 + \lambda_1^{-1}\right) b_3 = 0.$$

To get the last equality we used Lemma 3.11 twice, once with $y = c_1$ and $z = b_3$ and once with $y = c_1$ and $z = b_2$.

Similarly,

$$(a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1} b_2 = (a_{3,1} + a_{4,1} + a_{5,1}) \left(t_2 + \lambda_1^{-1}\right) b_2 = 0.$$

Notice that the previous two computations imply
$(a_{3,1} + a_{4,1} + a_{5,1}) t_0^{-1} c_3 = 0$ and $(a_{3,1} + a_{4,1} + a_{5,1}) t_0^{-1} (b_3 - c_3) = 0$.

Continuing our computation, we see the last expression in Equation 3.25 is equal to:

$$
\begin{aligned}
&1+\left(t_0^{-1}-t_0^{-1}\left(a_{3,1}+a_{4,1}+a_{5,1}\right)t_0^{-1}\right)\left(a_{3,2}+a_{4,2}+a_{4,3}+a_{5,2}+a_{5,3}\right)\\
&=1+\left(\left(t_2+\lambda_1^{-1}\right)-t_0^{-1}\left(a_{3,1}+a_{4,1}+a_{5,1}\right)t_0^{-1}\right)\left(a_{3,2}+a_{4,2}+a_{4,3}+a_{5,2}+a_{5,3}\right)\\
&=1+t_2a_{3,2}+t_2a_{4,2}+t_2a_{4,3}+t_2a_{5,2}+t_2a_{5,3}\\
&\qquad-t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}a_{3,2}\\
&\qquad-t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}(a_{4,2}+a_{4,3})\\
&\qquad-t_0^{-1}(a_{3,1}+a_{4,1}+a_{5,1})t_0^{-1}(a_{5,2}+a_{5,3})\\
&\qquad+\lambda_1^{-1}a_{3,2}+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3}+\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}\\
&=1+t_2a_{3,2}+t_2a_{4,2}+t_2a_{4,3}+\lambda_1^{-1}a_{3,2}\\
&\qquad+\lambda_1^{-1}a_{4,2}+\lambda_1^{-1}a_{4,3}+\lambda_1^{-1}a_{5,2}+\lambda_1^{-1}a_{5,3}.
\end{aligned}
\tag{3.26}
$$

Let $t_3=t_2a_{4,3}+1$. Notice that $t_3\in\overline{b_2Ab_2}+\mathbb{C}1_A$ since $t_2\in\overline{b_2Ab_2}$ and $a_{4,3}\in\overline{(b_3-c_3)Ab_2}$. Thus there is an invertible element $t_4\in\overline{b_2Ab_2}+\mathbb{C}1_A$ with

$$\|t_4-t_3\|<\delta. \tag{3.27}$$

Write $t_4=t_5+\lambda_5 1_A$ with $t_5\in\overline{b_2Ab_2}$ and $\lambda_5\in\mathbb{C}$. Similarly, write $t_4^{-1}=t_6+\lambda_5^{-1}1_A$ with $t_6\in\overline{b_2Ab_2}$.

Using the same argument used to show Equation 3.23 we can show that $\left(t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)^2=0$.

A similar computation to the one for Equation 3.24 gives

$$(1-t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}))t_4^{-1}=(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}+t_4)^{-1}. \tag{3.28}$$

Also notice $\left(1-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\right)t_4^{-1}=t_4^{-1}-t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)t_4^{-1}$.

Next we show $b_1t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)=0$. By applying Lemma 3.11, since $t_6\in\overline{b_2Ab_2}$, $b_1\in\overline{b_1Ab_1}$, and $a_{5,2}+a_{5,3}\in\overline{c_3A}$, we have

$$
\begin{aligned}
&b_1t_4^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\\
&=b_1t_6\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)+b_1\lambda_5^{-1}\left(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2}\right)\\
&=0.
\end{aligned}
\tag{3.29}
$$

Similarly, $b_2t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2})=0$. These two also imply that $(b_1-c_1)t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2})=0$ and $c_1t_4^{-1}(\lambda_1^{-1}a_{5,3}+\lambda_1^{-1}a_{5,2})=0$.

Now

$$
\begin{aligned}
&\left(t_2 a_{3,2} + t_2 a_{4,2} + t_4 + \lambda_1^{-1} a_{3,2} + \lambda_1^{-1} a_{4,2} + \lambda_1^{-1} a_{4,3} + \lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}\right) \\
&\qquad \cdot \left[\left(1 - t_4^{-1}\left(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}\right)\right) t_4^{-1}\right] \\
&= 1 + t_2 a_{3,2} t_4^{-1} + t_2 a_{4,2} t_4^{-1} + \lambda_1^{-1} a_{3,2} t_4^{-1} + \lambda_1^{-1} a_{4,2} t_4^{-1} + \lambda_1^{-1} a_{4,3} t_4^{-1} \\
&\qquad - t_2 a_{3,2} t_4^{-1}(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}) t_4^{-1} - t_2 a_{4,2} t_4^{-1}(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}) t_4^{-1} \\
&\qquad - \lambda_1^{-1} a_{3,2} t_4^{-1}\left(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}\right) t_4^{-1} - \lambda_1^{-1} a_{4,2} t_4^{-1}\left(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}\right) t_4^{-1} \\
&\qquad - \lambda_1^{-1} a_{4,3} t_4^{-1}\left(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}\right) t_4^{-1} \\
&= 1 + t_2 a_{3,2} t_4^{-1} + t_2 a_{4,2} t_4^{-1} + \lambda_1^{-1} a_{3,2} t_4^{-1} + \lambda_1^{-1} a_{4,2} t_4^{-1} + \lambda_1^{-1} a_{4,3} t_4^{-1} - 0 \\
&\qquad \text{by Equation 3.29 and the statements that follow it} \\
&= 1 + t_2 a_{3,2} t_6 + t_2 a_{4,2} t_6 + \lambda_1^{-1} a_{3,2} t_6 + \lambda_1^{-1} a_{4,2} t_6 + \lambda_1^{-1} a_{4,3} t_6 \\
&\qquad + t_2 a_{3,2} \lambda_5^{-1} + t_2 a_{4,2} \lambda_5^{-1} + \lambda_1^{-1} a_{3,2} \lambda_5^{-1} + \lambda_1^{-1} a_{4,2} \lambda_5^{-1} + \lambda_1^{-1} a_{4,3} \lambda_5^{-1}.
\end{aligned} \tag{3.30}
$$

Denote the quantity just computed by t_7 and notice that

$$
t_7 \in \overline{((b_1 - c_1) + b_2 + (b_3 + c_3))A((b_1 - c_1) + b_2 + (b_3 + c_3))} = \overline{c_2 A c_2},
$$

which has stable rank one by hypothesis.

Thus there exists an invertible element $t_8 \in \overline{c_2 A c_2} + \mathbb{C}1_A$ such that

$$
\|t_8 - t_7\| < \frac{\delta}{\left\|\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4\right\| + 1}. \tag{3.31}
$$

Now

$$
\begin{aligned}
&\left[t_0^{-1}\left(1 - (a_{3,1} + a_{4,1} + a_{5,1})\, t_0^{-1}\right)\right]^{-1} t_8 \left[\left(1 - t_4^{-1}\left(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3}\right)\right) t_4^{-1}\right]^{-1} \\
&= (a_{3,1} + a_{4,1} + a_{5,1} + t_0)\, t_8 \left(\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4\right)
\end{aligned}
$$

is invertible, and as we will now compute,

$$
\|a - (a_{3,1} + a_{4,1} + a_{5,1} + t_0) t_8 (\lambda_1^{-1} a_{5,2} + \lambda_1^{-1} a_{5,3} + t_4)\| < \varepsilon.
$$

We have

$$
\begin{aligned}
&\|a - (a_{3,1} + a_{4,1} + a_{5,1} + t_0)t_8(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4)\| \\
&= \|a_{3,1} + a_{3,2} + a_{3,3} + a_{4,1} + a_{4,2} + a_{4,3} + a_{5,1} + a_{5,2} + a_{5,3} \\
&\qquad - (a_{3,1} + a_{4,1} + a_{5,1} + t_0)t_8(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4)\| \\
&\leq \|a_{3,3} - t_0\| \\
&\qquad + \|(a_{3,1} + a_{4,1} + a_{5,1} + t_0)t_0^{-1}(1 - (a_{3,1} + a_{4,1} + a_{5,1})t_0^{-1}) \\
&\qquad\quad \cdot (a_{3,1} + a_{3,2} + t_0 + a_{4,1} + a_{4,2} + a_{4,3} + a_{5,1} + a_{5,2} + a_{5,3}) \\
&\qquad - (a_{3,1} + a_{4,1} + a_{5,1} + t_0)t_8(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4)\| \qquad \text{by Equation 3.24} \\
&\leq \delta + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad \cdot \|1 + t_2a_{3,2} + t_2a_{4,2} + t_2a_{4,3} + \lambda_1^{-1}a_{3,2} + \lambda_1^{-1}a_{4,2} + \lambda_1^{-1}a_{4,3} \\
&\qquad\quad + \lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} - t_8(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4)\| \\
&\qquad\qquad\qquad \text{by Equations 3.25 and 3.26} \\
&\leq \delta + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad\quad \cdot \|1 + t_2a_{3,2} + t_2a_{4,2} + t_2a_{4,3} + \lambda_1^{-1}a_{3,2} + \lambda_1^{-1}a_{4,2} + \lambda_1^{-1}a_{4,3} + \lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} \\
&\qquad\qquad - (t_2a_{3,2} + t_2a_{4,2} + t_4 + \lambda_1^{-1}a_{3,2} + \lambda_1^{-1}a_{4,2} + \lambda_1^{-1}a_{4,3} + \lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3})\| \\
&\qquad + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad\quad \cdot \|(t_2a_{3,2} + t_2a_{4,2} + t_4 + \lambda_1^{-1}a_{3,2} + \lambda_1^{-1}a_{4,2} + \lambda_1^{-1}a_{4,3} + \lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3}) \\
&\qquad\quad \cdot (1 - t_4^{-1}(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3}))t_4^{-1}(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4) \\
&\qquad\qquad - t_8(\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4)\| \quad \text{by Equation 3.28} \\
&\leq \delta + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\|\|1 + t_2a_{4,3} - t_4\| \\
&\qquad + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad\quad \cdot \|1 + t_2a_{3,2}t_4^{-1} + t_2a_{4,2}t_4^{-1} + \lambda_1^{-1}a_{3,2}t_4^{-1} + \lambda_1^{-1}a_{4,2}t_4^{-1} + \lambda_1^{-1}a_{4,3}t_4^{-1} - t_8\| \\
&\qquad\quad \cdot \|\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4\| \quad \text{by Equation 3.30} \\
&\leq \delta + \delta\|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \\
&\qquad + \|a_{3,1} + a_{4,1} + a_{5,1} + t_0\| \frac{\delta}{\|\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4\| + 1} \|\lambda_1^{-1}a_{5,2} + \lambda_1^{-1}a_{5,3} + t_4\| \\
&\qquad\qquad\qquad \text{by the choice of } t_4 \text{ and } t_8 \\
&\leq \delta + 2\delta\|a_{3,1} + a_{4,1} + a_{5,1} + a_{3,3}\| + 2\delta\|a_{3,3} - t_0\| \\
&< \varepsilon/2 + 2\varepsilon/6 + 2\delta^2 \\
&< \varepsilon.
\end{aligned}
$$

This completes the proof. ∎

Lemma 3.13 uses Lemma 3.12 to produce a simpler replacement for the decomposition of the identity into orthogonal projections.

Lemma 3.13. *Let A be a stably finite unital C^*-algebra, let $\varepsilon > 0$ be given, and let $x_1, x_2, x_3 \in A$ be positive elements such that $x_1 + x_2 + x_3 = 1$ and $x_1x_3 = 0$. Let $a \in A$ be such that*

$x_1 a = 0$, $a x_3 = 0$, and $\overline{x_2 A x_2}$ has stable rank 1. Then there exists an element $a_1 \in A$ such that a_1 is invertible and $\|a_1 - a\| < \varepsilon$.

Proof. It suffices to show that the hypotheses here imply the hypotheses of 3.12. Let $f : [0,1] \to [0,1]$ and $h : [0,1] \to [0,1]$ be defined by the formulas

$$f(t) = \begin{cases} 0 & t \in [0, \frac{1}{2}] \\ 2t - 1 & t \in [\frac{1}{2}, 1] \end{cases}$$

and

$$h(t) = \begin{cases} 2t & t \in [0, \frac{1}{2}] \\ 1 & t \in [\frac{1}{2}, 1]. \end{cases}$$

It is clear that $fh = f$. Now set $b_1 = h(x_1)$, $c_1 = f(x_1)$, $b_3 = h(x_3)$, $c_3 = f(x_3)$, $b_2 = f(x_2)$, and $c_2 = h(x_2)$, all of which are positive.

Note that the formulas $c_1 b_1 = c_1$, $c_3 b_3 = c_3$, and $c_2 b_2 = b_2$ all hold. Since $x_1 x_3 = 0$, we have $b_1 b_3 = h(x_1) h(x_3) = 0$, and so also $c_1 b_3 = 0 = b_1 c_3$. Similarly, $x_1 a = 0$ and $a x_3 = 0$ imply that $b_1 a = h(x_1) a = 0$ and $a b_3 = a h(x_3) = 0$, and these in turn imply that $c_1 a = 0$ and $a c_3 = 0$.

Also, $\overline{b_2 A b_2}$ and $\overline{c_2 A c_2}$ have stable rank one because they are hereditary subalgebras of $\overline{x_2 A x_2}$.

Then since $C^*(x_1, x_2, x_3)$ is commutative, $C^*(f(x_1), f(x_2), f(x_3), h(x_1), h(x_2), h(x_3))$ is commutative, hence is isomorphic to $C(Y)$ for some compact Hausdorff space Y. One can check that $c_1 + c_2 + c_3 = 1$ since c_1 and c_3 are orthogonal and $f(t) + h(t) = 1$ for all t. Similarly, $b_1 + b_2 + b_3 = 1$. ∎

Lemma 3.14. *Let A be a simple, unital C^*-algebra, and let $a, b \in A_+$ with $\|a\| = \|b\| = 1$. Then there exists $c \in A_+ \setminus \{0\}$ with $\|c\| \leq 1$ such that $c \leq a$ and $c \precsim b$.*

Proof. Since A is simple and $a, b \in A$ are nonzero, by Proposition 1.8 of [3] there is a nonzero $y \in A$ such that $yy^* \in \overline{aAa}$ and $y^* y \in \overline{bAb}$. Without loss of generality we may assume that $\|yy^*\| \leq 1$, and so $yy^* \leq 1$. Set $c = a^{1/2} y y^* a^{1/2}$. Set $z = (a^{1/2} y)^*$, and choose $0 < \beta < 1$. Then, $z^* z \leq z^* z$, so by Proposition 1.4.5 of [12] there is $u \in A$ such that $z = u (z^* z)^{\beta/2}$. Note that $[u (z^* z)^{\beta/2}] (z^* z)^{\beta/2} (u^*) = [u (z^* z)^{\beta/2}] [u (z^* z)^{\beta/2}]^* = z z^*$, and so $z z^* \precsim (z^* z)^{\beta/2}$. But since $f(t) = t$ and $g(t) = t^\beta$ are zero on the same set, $z^* z \sim (z^* z)^\beta$ by Lemma 2.4. Therefore $z z^* \precsim z^* z$. Symmetrically, $z^* z \precsim z z^*$. This implies $z z^* \sim z^* z$.

Note that $y^* a y \leq y^* y \in \overline{bAb}$, so $y^* a y \in \overline{bAb}$. Therefore, $y^* a y \precsim b$ by the second paragraph of section 1 in [4]. Combining this with $c = z^* z \sim z z^* = y^* a y$ yields $c \precsim b$. Furthermore, $yy^* \leq 1$ which implies $c = a^{1/2} y y^* a^{1/2} \leq a$. ∎

Lemma 3.15. *Let A be a simple unital infinite dimensional C^*-algebra. Let $a_1, a_2, a_3, a_4 \in A$ satisfy $a_i a_{i+1} = a_{i+1}$, for $i = 1, 2$, and 3, and $0 \leq a_1, a_2, a_3, a_4 \leq 1$. Also assume that at least one of a_1, a_2, and a_3 is not a projection, or that a_1, a_2 and a_3 are not all equal. Then $\tau(a_1) > \lim_{n \to \infty} \tau((a_4)^{1/n})$ for any tracial state τ on A.*

Proof. Notice that we have $a_i = a_{i-1}^{1/2} a_i a_{i-1}^{1/2} \leq a_{i-1}$ for $i = 2, 3$ or 4. Thus, $\tau(a_4) \leq \tau(a_3) \leq \tau(a_2) \leq \tau(a_1)$.

We first show that $\tau(a_1) > \tau(a_3)$. Since we have already observed that $\tau(a_1) \geq \tau(a_3)$, we only must show that they are not equal. Suppose $\tau(a_1) = \tau(a_3)$. Then $\tau(a_1 - a_3) = 0$. The hypotheses on A imply that τ is faithful, so $a_1 = a_3$.

But this means that $a_1a_2 = a_2$ and $a_1a_2 = a_3a_2 = a_3$, so $a_2 = a_3$ as well. If a_1, a_2, and a_3 are all distinct then this is a contradiction already. Otherwise, we now see $a_1 = a_2 = a_1a_2 = a_1^2$, so a_1 is a projection, but since all three are equal, we now see that a_2 and a_3 are also projections, which is a contradiction.

Now because $a_3a_4 = a_4$, we also have $a_3a_4^{1/n} = a_4^{1/n}$ for any n. Using a similar argument to the one used in the first paragraph this implies that $\tau(a_4^{1/n}) \leq \tau(a_3)$ for all n. Thus, $\lim_{n\to\infty} \tau(a_4^{1/n}) \leq \tau(a_3)$. Therefore, $\tau(a_1) > \tau(a_3) \geq \lim_{n\to\infty} \tau(a_4^{1/n})$. ∎

The following theorem is an analog of Lemma 5.2 of [11] with projections replaced by positive elements.

Lemma 3.16. *Let A be an infinite dimensional stably finite simple unital C^*-algebra. Suppose that every 2-quasi-trace on A is a trace. Also suppose A has finitely many extreme tracial states and that A has strict comparison. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Let $B = C^*(G, A, \alpha)$. Suppose $q_1, \ldots, q_n \in B$ are nonzero positive elements of norm at most one and $a_1, \ldots, a_m \in B$ are arbitrary. Let $\varepsilon > 0$ and $N \in \mathbb{N} \cup \{0\}$. Then there exist a subalgebra $D \subset B$ isomorphic to a matrix algebra over a hereditary subalgebra of A, a positive element $d \in D$ with $\|d\| \leq 1$, nonzero positive elements $r_{k,i} \in \overline{dDd}$ of norm at most 1 for $i = 0, \ldots, N$ and $k = 1, \ldots, n$, and elements $b_1, \ldots, b_m \in B$ such that the following conditions are satisfied.*

(1) $\|q_kr_{k,N} - r_{k,N}\| < \varepsilon$ *for all* $k = 1, \ldots, n$.
(2) $1 - d \precsim r_{k,N}$ *for all* $k = 1, \ldots, n$
(3) $r_{k,i}r_{k,i+1} = r_{k,i+1}$ *for all* $k = 1, \ldots, n$ *and* $i = 0, \ldots, N$.
(4) $r_{k,0}d = r_{k,0}$ *for all* $k = 1, \ldots, n$.
(5) $\|a_j - b_j\| < \varepsilon$ *for all* $j = 1, \ldots, m$.
(6) $db_jd \in \overline{dDd}$ *for all* $j = 1, \ldots, m$.

Proof. By rescaling, we may assume that $\|q_k\| = 1$ for $1 \leq k \leq n$. Let $\varepsilon_1 = \varepsilon/6$. Let $h_1 : [0, 1] \to [0, 1]$ be the continuous function which has $h_1(0) = 0$, $h_1(t) = 1$ for $t \in [1 - \varepsilon_1, 1]$, and is linear on $[0, 1 - \varepsilon_1]$. Let $h_2 : [0, 1] \to [0, 1]$ be the continuous function with $h_2(t) = 0$ for $t \in [0, 1 - \varepsilon_1]$, linear on $[1 - \varepsilon_1, 1]$, and $h_2(1) = 1$. Set $q_{j,1} = h_1(q_j)$ and $w_j = h_2(q_j)$. Note that $\|q_j - q_{j,1}\| \leq \varepsilon_1$. Set $\lambda = \min_{1\leq j\leq n} \inf_{\tau\in T(B)}\{\tau(w_j)\}$. Note that $\lambda \neq 0$ since $T(B)$ is compact and B simple implies $\tau(w_j) \neq 0$ for each τ and w_j.

If h is any continuous function which has $h(1) = 1$ and $0 \leq h \leq 1$, then $\tau(h(q_{j,1})) \geq \tau(w_j)$, thus we have

$$\tau(h(q_{j,1})) \geq \lambda. \tag{3.32}$$

Apply Lemma 3.10 with $\min\{\varepsilon/12, \lambda\}$ in place of δ to get a continuous function $g : [0, 1] \to [0, 1]$. Let $\varepsilon_2 < \lambda/4$. Now apply Lemma 2.12 with g as just obtained and with ε_2 in place of ε to get δ_2.

Let $\varepsilon_3 < \min\{\varepsilon/24, \delta_2, \lambda/8\}$. Then choose $\varepsilon_4 < \varepsilon_2$ such that if x, y are self adjoint elements of B, with $\|x\|, \|y\| \leq 1$ and $\|x - y\| < \varepsilon_4$, then $\|x_+ - y_+\| < \varepsilon_3$. Without loss of generality, we may assume that $\varepsilon_4 < \frac{\varepsilon}{2\max_j \|a_j\|+1}$.

Choose $\varepsilon_5 < \min\{\frac{\varepsilon}{12}, \frac{\varepsilon}{4\max_j \|a_j\|+1}, \frac{\lambda}{4}, \frac{\varepsilon_4}{4}\}$. Define the continuous function f_1 to be zero at zero, 1 on $[1 - \varepsilon_5, 1]$ and linear on $[0, 1 - \varepsilon_5]$. For $i = 2, \ldots, 5$, define f_i to be the continuous function which is zero on $[0, 1 - \varepsilon_5/(2^{i-2})]$, linear on

$[1-\varepsilon_5/(2^{i-2}), 1-\varepsilon_5/(2^{i-1})]$, and one on $[1-\varepsilon_5/(2^{i-1}), 1]$. Note that $f_1 f_2 = f_2$, $f_2 f_3 = f_3$, etc. and $\|f_1 - t\| < \varepsilon_5$.

Apply Lemma 2.13 with ε replaced by ε_3 and with f replaced by f_4, to get ε_6 such that $\|[x, y]\| < \varepsilon_6$ implies $\|[f_4(y), x]\| < \varepsilon_3$ if $\mathrm{sp}(y) \subset [0,1]$ and $\|x\| \le 1$.

Apply Lemma 3.9 with g as defined above and with ε_2 in place of ε to get ε_7.

Let $\mu_{\tau,j}$ be the measure obtained from τ on $\mathrm{sp}(q_j)$. For each extreme tracial state τ choose $\varepsilon_8(\tau, j)$ such that $\varepsilon_1/2 < \varepsilon_8(\tau, j) < \varepsilon_1$ and such that $\mu_{\tau,j}([1-\varepsilon_8(\tau, j), 1]) < \mu_{\tau,j}([1-\varepsilon_1/2, 1]) + \varepsilon_7$. Let
$\varepsilon_8 = \min\{\varepsilon_8(\tau, j) : \tau \text{ is extreme in } T(A) \text{ and } 1 \le j \le m\}$. Since there are only finitely many extreme tracial states, this minimum is achieved. Furthermore, note that $\varepsilon_1/2 < \varepsilon_8 < \varepsilon_1$ and

$$\mu([1-\varepsilon_8, 1]) < \mu([1-\varepsilon_1/2, 1]) + \varepsilon_7. \tag{3.33}$$

Define a continuous function h_3 such that $h_3(t) = 0$ for $t \in [0, 1-\varepsilon_8]$, $h_3(t) = 1$ for $t \in [1-\varepsilon_1/2, 1]$ and h_3 is linear on $[1-\varepsilon_8, 1-\varepsilon_1/2]$. Notice that $h_3 h_1 = h_3$. Let $q_{k,3} = h_3(q_k)$. Choose M with $\frac{1}{M} < \min\{\lambda/8 - \varepsilon_3, \varepsilon_7\}$.

Apply Lemma 3.4 with $F = \{q_{1,3}, \dots q_{n,3}, a_1, \dots, a_m\}$, with $\min\{\varepsilon_4/2, \varepsilon_6\}$ in place of ε, and with M in place of N, to obtain positive elements $c^{(1)} \in B$ and $c^{(1)}_{1,1} \in A$, a subalgebra $D \subset \overline{c^{(1)} B c^{(1)}}$ and an isomorphism $\Phi : M_n \otimes \overline{c^{(1)}_{1,1} A c^{(1)}_{1,1}} \to D$ such that there exist elements $x_1, \dots, x_n, e_1, \dots e_m \in D$ with

- $\|c^{(1)} q_{j,3} c^{(1)} - x_j\| < \varepsilon_4/2$ for all $j = 1, \dots n$ by part 3.
- $\|x_j\| \le \|q_{j,3}\| = 1$ for all $j = 1, \dots n$ by part 3.
- $\|c^{(1)} a_k c^{(1)} - e_k\| < \varepsilon_4/2$ for all $k = 1, \dots m$ by part 3.
- $\|c^{(1)} q_{j,3} c^{(1)} - x_j\| < \varepsilon_4/2$ for all $j = 1, \dots n$ by part 3.
- $\|c^{(1)} q_{j,3} - q_{j,3} c^{(1)}\| < \varepsilon_6$ for all $j = 1, \dots n$ by part 5, and
- $\tau(1 - c^{(1)}) < \frac{1}{M}$ by part 6.

Set $d^{(1)} = f_1(c^{(1)}), \dots, d^{(5)} = f_5(c^{(1)})$. We have $\|d^{(1)} - c^{(1)}\| < \varepsilon_5$. Also, $d^{(1)} d^{(2)} = d^{(2)}, \dots, d^{(4)} d^{(5)} = d^{(5)}$. Notice that since $1 - f_2(t)$ is zero on a larger set than $1 - t$, we have $1 - d^{(2)} \precsim 1 - c^{(1)}$ by Proposition 2.4. Similarly, we have

$$1 - d^{(2)} \precsim 1 - d^{(3)} \precsim 1 - d^{(4)} \precsim 1 - d^{(5)} \precsim 1 - c^{(1)}.$$

Now

$$\begin{aligned}
\|d^{(1)} q_{j,3} d^{(1)} - x_j\| &\le \|d^{(1)} q_{j,3} d^{(1)} - d^{(1)} q_{j,3} c^{(1)}\| + \|d^{(1)} q_{j,3} c^{(1)} - c^{(1)} q_{j,3} c^{(1)}\| \\
&\quad + \|c^{(1)} q_{j,3} c^{(1)} - x_j\| \\
&< 2\varepsilon_5 + \varepsilon_4/2.
\end{aligned}$$

Similarly, $\|d^{(1)} a_j d^{(1)} - e_j\| \le (2\varepsilon_5 + \varepsilon_4/2)\|a_j\|$.

Set $d = d^{(2)}$. Set $b_j = a_j + e_j - d^{(1)} a_j d^{(1)}$. Then

$$\|b_j - a_j\| = \|e_j - d^{(1)} a_j d^{(1)}\| \le (2\varepsilon_5 + \varepsilon_4/2)\|a_j\| < \left(\frac{2\varepsilon}{4\max_k \|a_k\| + 1} + \frac{\varepsilon}{4\max_k \|a_k\| + 1} \right) \|a_j\| \le \varepsilon.$$

Also, $d b_j d = d e_j d \in \overline{dDd}$. These are parts (5) and (6) of this lemma.

Notice that $\frac{1}{2}d^{(3)}(x_j + x_j^*)d^{(3)} \in \overline{d^{(3)}Dd^{(3)}} \subset \overline{dDd} \subset D$ is a self adjoint element of norm at most 1. We compute

$$\begin{aligned}\left\|d^{(3)}q_{j,3}d^{(3)} - \frac{1}{2}d^{(3)}(x_j + x_j^*)d^{(3)}\right\| &\le \left\|d^{(3)}d^{(1)}q_{j,3}d^{(1)}d^{(3)} - d^{(3)}x_jd^{(3)}\right\| \\ &\le \left\|d^{(1)}q_{j,3}d^{(1)} - x_j\right\| \\ &< 2\varepsilon_5 + \varepsilon_4/2 \\ &< \varepsilon_4.\end{aligned}$$

Thus, since $d^{(3)}q_{j,3}d^{(3)} \ge 0$, by the choice of ε_4, we see $y_j = (\frac{1}{2}d^{(3)}(x_j + x_j^*)d^{(3)})_+$ is a positive element of $\overline{dDd}$ with $\|d^{(3)}q_{j,3}d^{(3)} - y_j\| < \varepsilon_3$.

Now by the choice of g using Lemma 3.10 there exists a positive element of norm at most 1, $s_k \in \overline{y_kDy_k} \subset \overline{d^{(3)}Dd^{(3)}} \subset D$ such that

$$\|s_kg(y_k) - g(y_k)\| < \min\{\varepsilon/12, \lambda/4\} \text{ and } \|s_ky_k - s_k\| < \min\{\varepsilon/12, \lambda/4\}. \tag{3.34}$$

Let $r_k = s_kd^{(4)}$. Note that $r_k \in \overline{d^{(3)}Dd^{(3)}}$, so

$$dr_k = d^{(2)}r_k = r_k. \tag{3.35}$$

Using the choice of ε_6 in the last step since $\|q_{k,3}c^{(1)} - c^{(1)}q_{k,3}\| < \varepsilon_6$, we see that

$$\begin{aligned}\|r_kq_{k,3} - r_k\| \le\ &\|s_kd^{(4)} - s_ky_kd^{(4)}\| + \|s_ky_kd^{(4)} - s_kd^{(3)}q_{k,3}d^{(3)}d^{(4)}\| \\ &+ \|s_kd^{(3)}q_{k,3}d^{(3)}d^{(4)} - s_kd^{(3)}d^{(4)}q_{k,3}\| + \|s_kd^{(3)}d^{(4)}q_{k,3} - s_kd^{(4)}q_{k,3}\| \\ <\ &\varepsilon/12 + 2\varepsilon_3.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|r_kq_k - r_k\| &\le \|r_kq_k - r_kq_{k,1}\| + \|r_kq_{k,1} - r_kq_{k,3}q_{k,1}\| + \|r_kq_{k,3}q_{k,1} - r_k\| \\ &< \varepsilon_1 + 2(\varepsilon/12 + 2\varepsilon_3) \\ &< \varepsilon/2. \end{aligned} \tag{3.36}$$

Define h to be the continuous function which is 0 on $[0, \varepsilon_3]$, linear on $[\varepsilon_3, 1]$, and 1 at $t = 1$. Notice $\|h(t) - t\| < \varepsilon_3$ so $\|r_k - h(r_k)\| < \varepsilon_3$. Now define continuous functions $H_k = H_{N-j}$ for $j = 1, \ldots, N$ by H_{N-j} is 0 on $[0, \varepsilon_3(1 - \frac{j}{N})]$, linear on $[\varepsilon_3(1 - \frac{j}{N}), \varepsilon_3(1 - \frac{j-1}{N})]$ and 1 on $[\varepsilon_3(1 - \frac{j-1}{N}), 1]$. Set $H_N = h$. Notice that $H_jH_{j+1} = H_{j+1}$.

Define $r_{k,j} = H_j(r_k)$. Thus, $r_{k,j}r_{k,j+1} = r_{k,j+1}$. This is part (3) of the lemma. Now, by Equation 3.35, we have $dr_k = r_k$, so since $H_N(0) = 0$, we also get $dr_{k,N} = r_{k,N}$ which is part (4) of the lemma.

To obtain part (1) of the lemma we compute

$$\begin{aligned}\|r_{k,N}q_k - r_{k,N}\| &\le \|r_{k,N}q_k - r_kq_k\| + \|r_kq_k - r_k\| + \|r_k - r_{k,N}\| \\ &\le 2\varepsilon_3 + \frac{\varepsilon}{2} \quad \text{by equation 3.36} \\ &< \varepsilon.\end{aligned}$$

Thus (1) is proved.

It remains only to prove part (2). Since A and hence D have strict comparison we will begin by looking at traces.

We observe that

$$(1-d^{(k)})(1-d^{(k-1)}) = 1-d^{(k)}-d^{(k-1)}+d^{(k)}d^{(k-1)} = 1-d^{(k-1)}. \tag{3.37}$$

Now for any $\tau \in T(A)$,

$$\begin{aligned}
\tau\left(\left(1-d^{(4)}\right) g(y_k)s_k \left(1-d^{(5)}\right)\right) &= \tau\left(\left(1-d^{(5)}\right)\left(1-d^{(4)}\right) g(y_k)s_k\right) \\
&= \tau\left(\left(1-d^{(4)}\right)^{1/2} g(y_k)s_k \left(1-d^{(4)}\right)^{1/2}\right) \\
&\leq \tau\left(1-d^{(4)}\right) \\
&\leq \tau\left(1-c^{(1)}\right) \\
&< \frac{1}{M}.
\end{aligned}$$

But on the other hand

$$\begin{aligned}
\tau\left(\left(1-d^{(4)}\right) g(y_k)s_k \left(1-d^{(5)}\right)\right) &= \tau\left(g(y_k)s_k - d^{(4)}g(y_k)s_k - g(y_k)s_kd^{(5)} + g(y_k)s_kd^{(5)}d^{(4)}\right) \\
&= \tau\left(g(y_k)s_k - g(y_k)s_kd^{(4)}\right) \\
&= \tau\left(g(y_k)s_k - g(y_k)r_k\right).
\end{aligned}$$

Therefore,

$$\tau(g(y_k)s_k - g(y_k)r_k) < \frac{1}{M}. \tag{3.38}$$

If $z \in \overline{q_{k,3}Dq_{k,3}}$ and $\|z\| \leq 1$, then $\tau(z) \leq \mu([1-\varepsilon_8, 1]) < \mu([1-\varepsilon_1/2, 1]) + \varepsilon_7$ by Equation 3.33. Thus

$$\|\tau|_{\overline{q_{j,3}Dq_{j,3}}}\| - \varepsilon_7 < \tau(q_{j,3}). \tag{3.39}$$

Next we get a lower bound on $\tau(r_{k,N})$. We have

$$\begin{aligned}
\tau(r_{k,N}) &> \tau(r_k) - \varepsilon_3 \\
&\geq \tau\left(g(y_k)r_k\right) - \varepsilon_3 \\
&> \tau\left(g(y_k)s_k\right) - 1/M - \varepsilon_3 \qquad \text{by 3.38} \\
&> \tau\left(g(y_k)\right) - \lambda/4 - 1/M - \varepsilon_3 \qquad \text{by 3.34} \\
&> \tau\left(g\left(d^{(3)}q_{j,3}d^{(3)}\right)\right) - \varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \quad \text{since } \|d^{(3)}q_{j,3}d^{(3)} - y_j\| < \varepsilon_3 < \delta_2.
\end{aligned}$$

We can improve on this, because Equation 3.39 and $\tau(1-d^{(3)}) \leq \tau(1-c^{(1)}) < 1/M < \varepsilon_7$ together imply $\tau\left(g\left(d^{(3)}q_{j,3}d^{(3)}\right)\right) > \tau(q_{j,3}) - \varepsilon_2$ by the choice of ε_7 using Lemma 3.9. So now we get

$$\begin{aligned}
\tau(r_{k,N}) &> \tau\left(g\left(d^{(3)}q_{j,3}d^{(3)}\right)\right) - \varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \\
&> \tau\left(q_{j,3}\right) - \varepsilon_2 - \varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \\
&\geq \lambda - 2\varepsilon_2 - \lambda/4 - 1/M - \varepsilon_3 \text{ by Equation 3.32 with } \ h_3 \text{ in place of } h \\
&> \lambda/8 - \varepsilon_3 \\
&> 1/M \\
&> \tau\left(1 - c^{(1)}\right) \\
&> \tau\left(1 - d^{(5)}\right).
\end{aligned}$$

If at least one of $1 - d^{(3)}$, $1 - d^{(4)}$, and $1 - d^{(5)}$ is not a projection, then we have

$$\tau\left(1 - d^{(5)}\right) \geq \lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right)$$

for every $\tau \in T(A)$ by Lemma 3.15.

We can reach the same conclusion if all three of them are projections. First notice that by definition of the functions, $0 \leq 1 - f_2(t) \leq 1 - f_5(t) \leq 1$ for all t. But this implies that $0 \leq 1 - f_2\left(c^{(1)}\right) \leq 1 - f_5\left(c^{(1)}\right) \leq 1$, which means that $0 \leq 1 - d^{(2)} \leq 1 - d^{(5)} \leq 1$. By using Proposition 1.3.8 of [12] to get the inequality and the fact that $1 - d^{(5)}$ is a projection to get the equality we see that $\left(1 - d^{(2)}\right)^{1/n} \leq \left(1 - d^{(5)}\right)^{1/n} = 1 - d^{(5)}$ for any positive integer n. Therefore, $\lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right) \leq \tau\left(1 - d^{(5)}\right)$.

Either way, combining the estimate on $\tau(1 - d^{(5)})$ and the estimate on $\tau(r_{k,N})$ gives

$$\tau\left(r_{k,N}\right) > \tau\left(1 - d^{(5)}\right) \geq \lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right).$$

This implies

$$\lim_{n\to\infty} \tau\left(\left(r_{k,N}\right)^{1/n}\right) > \lim_{n\to\infty} \tau\left(\left(1 - d^{(2)}\right)^{1/n}\right)$$

for every $\tau \in T(A)$. Because A, and hence D, has strict comparison, we now can conclude that $1 - d^{(2)} \precsim r_{k,N}$ which is part (2) of the lemma. ∎

The following theorem is the main theorem of the article. It is a projection free, finite group analog of Theorem 5.3 of [11].

Theorem 3.17. *Let A be an infinite dimensional stably finite simple unital C^*-algebra such that all 2-quasi-traces are traces, and with only finitely many extreme tracial states. Assume A has stable rank one and strict comparison. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group with the projection free tracial Rokhlin property. Then $B = C^*(G, A, \alpha)$ also has stable rank one.*

Proof. Note that B has a faithful tracial state, so every one sided invertible element is invertible. Now, Theorem 3.3 (a) of [19] states that if the two sided zero divisors of B are contained in the closure of the invertible elements, then the complement of the invertible elements consists of those elements of B which are one sided, but not two sided invertible. Combining these two statements would give $B \setminus \overline{GL(B)} = \emptyset$

which means B has stable rank one. Therefore, it is sufficient to prove that for every two sided zero divisor $a \in B$ and every $\varepsilon > 0$, there is an invertible element of B within ε of a. Without loss of generality, $\|a\| \leq 1/2$ and $\varepsilon \leq 1$.

Now suppose $x, y \in B$ are nonzero and satisfy $xa = ay = 0$. Since $\|x^*x\|^{-1}x^*xa = ayy^*\|yy^*\|^{-1} = 0$ we may assume that x and y are positive elements of norm 1.

Let $\delta_1 = \min\left\{\frac{\varepsilon}{28}, \frac{\sqrt{\varepsilon}}{2\sqrt{11}}, \frac{1}{14}\right\}$. Apply Lemma 3.16 to the positive elements x and y in place of $q_1, \ldots, q_n$ and the element a in place of $a_1, \ldots, a_m$, with $N = 1$ and with δ_1 in place of ε. Call the resulting subalgebra A_0. Let p_0 be the resulting positive element d. Let $x_{0,0}, x_{0,1}, y_{0,0}$, and $y_{0,1}$ be the nonzero positive elements of norm one $r_{k,i}$. Let a_0 be the resulting element b_1.

Define $a_1 = (1 - x_{0,0})a_0(1 - y_{0,0})$. Note that $x_{0,1}a_1 = (x_{0,1} - x_{0,1}x_{0,0})a_0(1 - y_{0,0}) = 0$ and similarly, $a_1y_{0,1} = 0$. Next we wish to show that a_1 is near a. Since $\|a\| \leq 1/2$, we have

$$\begin{aligned}\|x_{0,0}a_0\| &\leq \|x_{0,0}\|\|a_0 - a\| + \|x_{0,0} - x_{0,0}x\|\|a\| + 0 \\ &\leq 2\delta_1.\end{aligned}$$

Similarly $\|a_0y_{0,0}\| < 2\delta_1$.

Now we can compute

$$\begin{aligned}\|a - a_1\| &\leq \|a - a_0\| + \|a_0 - (1 - x_{0,0})a_0(1 - y_{0,0})\| \\ &\leq \delta_1 + \|a_0y_{0,0}\| + \|x_{0,0}a_0\|\|1 - y_{0,0}\| \\ &\leq 7\delta_1.\end{aligned}$$

Now apply Lemma 3.14 with $x_{0,1}$ in place of b, and $y_{0,1}$ in place of a. From this lemma we get a positive nonzero element r of norm at most 1 with $r \leq x_{0,1}$ and $r \precsim y_{0,1}$.

Choose $\delta_2 < 2\delta_1$. Since A_0 is is stably isomorphic to A, Theorem 3.6 in [18] implies that the stable rank of A_0 is one. Thus for f_{δ_2} as defined in Definition 2.2, by Proposition 2.3 there exists a unitary $v \in U(A_0^+)$ such that $v^*f_{\delta_2}(r)v \in \overline{y_{0,1}Ay_{0,1}}$ where A_0^+ is the unitization of A_0. Set $r_1 = f_{\delta_2}(r)$.

Next we prove that a_1v^* is a zero divisor. We have

$$\|a_1v^*r_1\| = \lim_{n\to\infty} \|a_1y_{0,1}^{1/n}v^*r_1v\| = 0.$$

Therefore, $(a_1v^*)r_1 = 0$. On the other side we see that, since $r \leq x_{0,1}$, the elements r and thus r_1 are in the hereditary subalgebra generated by $x_{0,1}$, so

$$r_1(a_1v^*) = \lim_{n\to\infty} r_1(x_{0,1})^{1/n}a_1v^* = 0.$$

Let $\delta_3 = \min\{\frac{\varepsilon}{22}, \frac{\sqrt{\varepsilon}}{2\sqrt{11}}\}$ Apply Lemma 3.16 with the positive element of norm one r_1 in place of $q_1, \ldots q_n$, and with a_1v^* in place of $a_1, \ldots, a_m$. Use δ_3 in place of ε and $N = 1$. Call the resulting algebra A_2. Let p_2 be the resulting positive element of A_2, Let the resulting positive elements $r_{k,i}$ of norm at most 1 be called $x_{2,0}$ and $x_{2,1}$, and let the resulting element b_j be called a_2.

Define $a_3 = (1 - x_{2,0})a_2(1 - x_{2,0})$. Then $x_{2,1}a_3 = a_3x_{2,1} = 0$. Next we compute the norm of $\|a_2\|$. We have

$$\begin{aligned}\|a_2\| &\leq \|a_2 - a_1v^*\| + \|a_1v^* - av^*\| + \|av^*\| \\ &\leq \delta_3 + 7\delta_1 + 1/2.\end{aligned}$$

Now in order to estimate $\|a_2 - a_3\|$ we bound $\|x_{2,0}a_2\|$. We have

$$\begin{aligned}\|x_{2,0}a_2\| &\leq \|x_{2,0}a_2 - x_{2,0}a_1v^*\| + \|x_{2,0}a_1v^* - x_{2,0}r_1a_1v^*\| + \|x_{2,0}r_1a_1v^*\| \\ &\leq \delta_3 + \delta_3\|a_1\| \\ &\leq \frac{3\delta_3}{2} + 7\delta_3\delta_1.\end{aligned}$$

Similarly, $\|a_2x_{2,0}\| \leq \frac{3\delta_3}{2} + 7\delta_3\delta_1$.

Next we can estimate $\|a_2 - a_3\|$. We have

$$\begin{aligned}\|a_2 - a_3\| &= \|a_2 - (1 - x_{2,0})a_2(1 - x_{2,0})\| \\ &\leq \|a_2x_{2,0} + x_{2,0}a_2 - x_{2,0}a_2x_{2,0}\| \\ &\leq \|a_2x_{2,0}\| + \|x_{2,0}a_2\|\|1 - x_{2,0}\| \\ &\leq \frac{3\delta_3}{2} + 7\delta_3\delta_1 + 2\left[\frac{3\delta_3}{2} + 7\delta_3\delta_1\right] \\ &= \frac{9\delta_3}{2} + 21\delta_1\delta_3.\end{aligned}$$

The conclusion of Lemma 3.16 gives us that $x_{2,0}p_2 = x_{2,0}$. Thus

$$p_2a_3p_2 = p_2^{1/2}(1 - x_{2,0})\left(p_2^{1/2}a_2p_2^{1/2}\right)(1 - x_{2,0})\,p_2^{1/2}.$$

Now $(p_2^{1/2}a_2p_2^{1/2}) \in \overline{p_2A_2p_2}$, and $1 - x_{2,0} \in \overline{p_2A_2^+p_2}$. Therefore, $p_2a_3p_2 \in \overline{p_2A_2^+p_2}$.

With f_{δ_4} as defined in 2.2, choose δ_4 so that $f_{\delta_4}(1 - p_2) \neq 0$. Note that this is possible unless $\mathrm{sp}(1 - p_2) = \{0\}$ in which case $p_2 = 1$. If this occurs, then $p_2A_2p_2 = A_2$ which has stable rank one. Then we can approximate a_2 by an invertible element and be finished with the proof. Therefore, we may assume that we can choose such a δ_4.

By the conclusion of Lemma 3.16, we have $1 - p_2 \preccurlyeq x_{2,1}$. Thus by Proposition 2.3 there exists a unitary $u \in U(A_2^+)$ such that $uf_{\delta_4}(1 - p_2)u^* \in \overline{x_{2,1}A_2^+x_{2,1}}$. Then, since $x_{2,0}x_{2,1} = x_{2,1}$ and $uf_{\delta_4}(1 - p_2)u^* \in \overline{x_{2,1}A_2^+x_{2,1}}$, we have

$$x_{2,0}uf_{\delta_4}(1 - p_2)u^* = uf_{\delta_4}(1 - p_2)u^* = uf_{\delta_4}(1 - p_2)u^*x_{2,0}.$$

Thus

$$uf_{\delta_4}(1 - p_2)u^*(a_3u) - uf_{\delta_4}(1 - p_2)u^*(1 - x_{2,0})a_2(1 - x_{2,0})u = 0$$

and similarly, $(a_3u)f_{\delta_4}(1 - p_2)u^* = 0$. This implies $a_3uf_{\delta_4}(1 - p_2) = 0$.

Next we observe that $uf_{\delta_4}(1 - p_2)u^*$ and $f_{\delta_4}(1 - p_2)$ are orthogonal. First, using $x_{2,0}x_{2,1} = x_{2,1}$ again we see

$$[u(1 - p_2)u^*](1 - p_2) = u(1 - p_2)u^*(x_{2,0} - x_{2,0}p_2) = 0.$$

Therefore, for any continuous function f with $f(0) = 0$, we have $uf(1 - p_2)u^*$ is orthogonal to $f(1 - p_2)$. In particular, $uf_{\delta_4}(1 - p_2)u^*$ is orthogonal to $f_{\delta_4}(1-p_2)$. Set $x_1 = uf_{\delta_4}(1-p_2)u^*$ and $x_3 = f_{\delta_4}(1-p_2)$. Set $x_2 = 1-x_1-x_3$. Note that $0 \leq x_2 \leq 1$. Our goal now is to use Lemma 3.13 with these choices of x_1, x_2, and

x_3 and with a replaced by a_3u. We have already shown that $x_1a_3u = a_3ux_3 = 0$. We must show that $x_2 \in p_2A_2^+p_2$.

First we show that $1 - f_{\delta_4}(1-p_2) \in \overline{p_2A_2p_2}$. Observe that $1-(1-p_2) = p_2 \in \overline{p_2A_2p_2}$. Also, since 1 and p_2 commute, using the binomial expansion theorem, we can show that $1-(1-p_2)^n \in \overline{p_2A_2p_2}$. In fact for any polynomial with $f(0) = 0$ and $f(1) = 1$, we have $1 - f(1-p_2) \in \overline{p_2A_2p_2}$. Since f_{δ_4} is the limit of such polynomials, $1 - f_{\delta_4}(1-p_2) \in \overline{p_2A_2p_2}$.

Next recall that $uf_{\delta_4}(1-p_2)u^* \in \overline{x_{2,1}A_2^+x_{2,1}} \subset A_2^+$. Additionally,

$$\lim_{n\to\infty} p_2^{1/n}uf_{\delta_4}(1-p_2)u^* = \lim_{n\to\infty}\lim_{m\to\infty} p_2^{1/n}x_{2,1}^{1/m}uf_{\delta_4}(1-p_2)u^* = uf_{\delta_4}(1-p_2)u^*.$$

A similar computation works on the other side, so we see that $uf_{\delta_4}(1-p_2)u^* \in \overline{p_2A_2^+p_2}$. This implies that $x_2 = 1 - uf_{\delta_4}(1-p_2)u^* - f_{\delta_4}(1-p_2) \in \overline{p_2A_2^+p_2}$ which has stable rank one because A_2 is isomorphic to matrices over a hereditary subalgebra of A.

Now we may apply Lemma 3.13 with x_1, x_2, and x_3 as above, with A replaced by A_2^+, with a_3u in place of a, and with $\varepsilon/44$ in place of ε. The lemma gives us an invertible element $a_4 \in A_2^+$ with $\|a_4 - a_3u\| < \varepsilon/44$. Then a_4u^*v is invertible and near a. More specifically,

$$\begin{aligned}\|a_4u^*v - a\| &\le \|a_4u^*v - a_3v\| + \|a_3v - a_2v\| + \|a_2v - a_1\| + \|a_1 - a\| \\ &< \varepsilon/44 + \frac{9\delta_3}{2} + 21\delta_1\delta_3 + \delta_3 + 7\delta_1 \\ &< \varepsilon.\end{aligned}$$

Therefore, $C^*(G, A, \alpha)$ has stable rank one. ∎

4. COMMENTS ABOUT THE DEFINITION

In the cases under consideration in this paper, the definition given for the projection free tracial Rokhlin property worked well. However, in more general contexts more conditions may be needed. This is similar to the situation which occurred in the presence of projections. See, for example, Lemma 1.12 of [17].

Definition 4.1. *Let A be an infinite dimensional unital simple C^*-algebra. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. We say α has the* strong form of the projection free tracial Rokhlin property *if for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there exist mutually orthogonal elements $a_g \in A$ for each $g \in G$ with $0 \le a_g \le 1$ such that:*

(1) $\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ for all $g, h \in G$.
(2) $\|a_gb - ba_g\| < \varepsilon$ for all $g \in G$ and $b \in F$.
(3) With $a = \sum_{g\in G} a_g$, the element $1 - a$ is Cuntz subequivalent to an element of the hereditary subalgebra generated by x.
(4) $\|axa\| > 1 - \varepsilon$.
(5) $\tau(1 - a) < \varepsilon$ for all $\tau \in T(A)$.

Note that since any element of $\overline{xAx}$ is subequivalent to x, the third condition is equivalent to $1 - a \precsim x$.

Proposition 4.2. *If A is an infinite dimensional unital stably finite simple C^*-algebra and $\alpha : G \to \mathrm{Aut}(A)$ is an action of a finite group with the strong form of the projection free tracial Rokhlin property as given above, then α has the projection free tracial Rokhlin property as given in Definition 2.7.*

Remark 4.3. *If we have an action with the strong form of the projection free tracial Rokhlin property, then in Lemma 3.3 we can also arrange for $1 - a \precsim x$ and $\|axa\| > 1 - \delta$. This is easy to see from the proof by adding in the words "strong form of" before both applications of the projection free tracial Rokhlin property.*

The content of the following lemma is that the condition $1 - a \precsim x$, in the projection free tracial Rokhlin property (or its strong form), is not needed when the algebra has strict comparison and every 2-quasi-trace is a trace.

Proposition 4.4. *Let A be an infinite dimensional unital simple C^*-algebra. Suppose also that all 2-quasi-traces on A are traces and that A has strict comparison. Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a finite group G on A. Suppose that for every finite set $F \subset A$, every $\varepsilon > 0$, and every positive element $x \in A$ with $\|x\| = 1$, there exist mutually orthogonal elements $a_g \in A$ for each $g \in G$ with $0 \leq a_g \leq 1$ such that:*

(1) $\|\alpha_g(a_h) - a_{gh}\| < \varepsilon$ *for all* $g, h \in G$.
(2) $\|a_g b - b a_g\| < \varepsilon$ *for all* $g \in G$ *and* $b \in F$.
(3) With $a = \sum_{g \in G} a_g$, *we have* $\|axa\| > 1 - \varepsilon$.
(4) $\tau(1 - a) < \varepsilon$ *for all* $\tau \in T(A)$.

Then α has the strong form of projection free tracial Rokhlin property and hence, if A is also stably finite, the projection free tracial Rokhlin property.

5. EXAMPLE

Remark 5.1. *Let Z be the Jiang-Su algebra as constructed in* [7]*. Then Z is an infinite dimensional stably finite exact simple unital C^*-algebra with a unique trace. Also, Z has strict comparison and stable rank one. In other words, Z satisfies the hypotheses on the algebra in Theorem 3.17.*

Proof. By Theorem 2.9 of [7] Z is infinite dimensional, unital and simple with a unique tracial state. The algebra Z is stably finite since it has a faithful tracial state. We see that Z is exact because it is the direct limit of exact C^*-algebras. By Theorem 5.3 of [21], Z has strict comparison because it is simple and it is the inductive limit of a sequence of recursive subhomogeneous algebras with slow dimension growth. Because all the maps in the system are unital, an $\varepsilon/3$ argument shows that Z has stable rank one. ∎

Recall that $Z \cong Z \otimes Z$.

Example 5.2. *Let $G = \mathbb{Z}/2\mathbb{Z} = \{1, -1\}$. Let $\alpha : G \to \mathrm{Aut}(Z \otimes Z)$ be given by $\alpha_{-1} : Z \otimes Z \to Z \otimes Z$ maps $a \otimes b$ to $b \otimes a$. Then α has the projection free tracial Rokhlin property.*

Before proving the action in Example 5.2 has the projection free tracial Rokhlin property we will need some lemmas.

The following is a proposition in [13] and Z satisfies its hypotheses [14].

Proposition 5.3. *Let A be a unital C^*-algebra. Let τ be a tracial state on A such that, with π_τ being the associated Gelfand-Naimark-Segal representation, the von Neumann algebra $\pi_\tau(A)''$ has no minimal projections. Let $S = \{a \in A \colon 0 \leq a \leq 1\}$. For $a \in S$ let μ_a be the Borel probability measure on $[0,1]$ defined by $\int_0^1 f \, d\mu_a = \tau(f(a))$ for $f \in C([0,1])$. Then there exists a dense G_δ-set $G \subset S$ such that, for every $a \in G$ and every $t \in [0,1]$, we have $\mu_a(\{t\}) = 0$.*

Lemma 5.4. *Let $X = [0,1]^2$. Let $h : X \to X$ be the flip given by $h(s,t) = (t,s)$. Suppose μ_0 is a Borel probability measure on $[0,1]$ such that $\mu_0(\{t\}) = 0$ for all $t \in [0,1]$. Define μ to be the product measure $\mu_0 \times \mu_0$ on X. Then for every $\varepsilon > 0$, there exists a closed subset $Y \subset X$ such that*

(1) $Y \cap h(Y) = \emptyset$
(2) $\mu[X \setminus (Y \cup h(Y))] < \varepsilon$

Proof. Define $E = \{(s,t) \in X : s < t\}$ and $\Delta = \{(s,t) \in X : s = t\}$. Fubini's Theorem and the fact that $\mu_0(\{t\}) = 0$ for all $t \in [0,1]$ imply $\mu(\Delta) = 0$. Note that the homeomorphism h preserves μ. Therefore, $\mu(E) = \mu(h(E))$. Since $E \cap h(E) = \emptyset$ we have

$$\begin{aligned} 1 &= \mu(X \setminus \Delta) \\ &= \mu(E \cup h(E)) \\ &= \mu(E) + \mu(h(E)) \\ &= 2\mu(E). \end{aligned}$$

Therefore, $\mu(E) = \frac{1}{2}$.

By the inner regularity of μ choose a compact set $Y \subset E$ such that $\mu(Y) > \mu(E) - \frac{\varepsilon}{2}$.

Then we have

$$\begin{aligned} \mu(X \setminus [Y \cup h(Y)]) &= 1 - \mu(Y) - \mu(h(Y)) \\ &= 1 - 2\mu(Y) \\ &\leq 1 - 2(\mu(E) - \varepsilon/2) \\ &= \varepsilon. \end{aligned}$$

This completes the proof. ∎

Now we are in a position to prove the statement in Example 5.2.

Proof of Example 5.2. Let $\varepsilon > 0$ and a finite subset $F \subset Z \otimes Z$ be given. Recall that

$$Z \otimes Z \cong \varinjlim (Z \otimes Z)^{\otimes n}$$

with connecting maps given by $z \mapsto z \otimes 1$. Therefore, there exists N sufficiently large such that for every $d \in F$, there is some $e \in (Z \otimes Z)^{\otimes N}$ with $\|d - e\| < \varepsilon/3$.

For any positive $y \in Z$, we have $\mathrm{sp}(y)$ is connected since there are no notrivial projections. In particular, $\mathrm{sp}(y)$ is either a point or an interval, but the only elements which have a single point for their spectrum are the scalars. Furthermore, if μ_y is defined as in Proposition 5.3, and $\mathrm{sp}(y)$ is a point, then $\mu_y(\{t\}) = 0$ for all $t \neq y$. Therefore, every element of the dense G_δ set found in 5.3 has an interval for its spectrum. Choose y out of this dense set and by rescaling we may assume that $\mathrm{sp}(y) = [0,1]$. Let $\mu_0 = \mu_y$.

Apply Lemma 5.4 with ε and μ_0 as given to obtain a closed subset $Y \subset [0,1]^2$. As in Lemma 5.4 let μ be the product measure of μ_0 with itself on $[0,1]^2$. Let ω be the corresponding tracial state on $C([0,1]^2)$. Let $U \subset [0,1]^2$ be an open subset satisfying $U \cap h(U) = \emptyset$ and $Y \subset U$, where h is the flip as defined in the statement of 5.4. Let $\gamma : C([0,1]^2) \to C([0,1]^2)$ be defined by $g \mapsto g \circ h$.

Let f be a continuous function which is 1 on Y and has $\mathrm{supp}(f) \subset U$. Define $\phi_0 : C([0,1]) \to Z$ by $\phi_0(g) = g(y)$. Let $\phi : C([0,1]^2) \to (Z \otimes Z)^N \otimes (Z \otimes Z)$ be given by $g \mapsto 1 \otimes (\phi_0 \otimes \phi_0)(g)$. Note that $\phi \circ \gamma = \alpha_{-1} \circ \phi$ and $\tau \circ \phi = \omega$.

Set $b_1 = f$ and $b_{-1} = \gamma(f)$. Then set $a_1 = \phi(b_1)$ and $a_{-1} = \phi(b_{-1})$. It remains to prove that these are the desired elements. Note that a_1 and a_{-1} are orthogonal since f and $\gamma(f)$ are orthogonal by virtue of $U \cap h(U) = \emptyset$. Similarly, they are positive because f and $\gamma(f)$ are positive and so is y. Note $\|a_g\| = \|\phi(b_g)\| = \|b_g\| = \|f\| = 1$.

For (1) we have $\alpha_{-1}(a_1) = \alpha_{-1}(\phi(f)) = \phi(\gamma(f)) = \phi(b_{-1}) = a_{-1}$. Similarly, $\alpha_{-1}(a_{-1}) = a_1$.

By the choice of N, for each $d \in F$ there exists $e \in (Z \otimes Z)^{\otimes N}$ with $\|d - e\| < \varepsilon/3$. Since $a_g \in 1 \otimes (Z \otimes Z)$ we have $\|a_g d - d a_g\| \leq \|a_g d - a_g e\| + \|a_g e - e a_g\| + \|e a_g - d a_g\| < 2\varepsilon/3$. This shows that (2) holds.

Condition (3) is shown by computing

$$\begin{aligned} \tau(1-a) &= \tau\left(\phi\left(1 - f - \gamma\left(f\right)\right)\right) \\ &= \omega\left(1 - f - \gamma\left(f\right)\right) \\ &= \int_{[0,1]^2} \left(1 - f - \gamma\left(f\right)\right) d\mu \\ &< \mu\left([0,1]^2 \setminus \left(Y \cup h\left(Y\right)\right)\right) \\ &< \varepsilon. \end{aligned}$$

Therefore, the tensor flip on $Z \otimes Z$ has the projection free tracial Rokhlin property. ∎

Corollary 5.5. *Let G and α be as in Example 5.2. Then $C^*(G, Z, \alpha)$ has stable rank one.*

Proof. This is an immediate consequence of Remark 5.1, Example 5.2, and Theorem 3.17. ∎

The fact that $C^*(G, Z, \alpha)$ has stable rank one is not new (see Theorem 1.1 of [6]), although the above proof is different.

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